Solutions of a system of functional equations in special class functions

By S. CZERWIK (Katowice)

1. In the present paper we consider the problem of the existence, uniqueness and continuous dependence on parameter u of solutions of the system of functional equations

$$\varphi_i(x) = h_i(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_n[f_{i,n}(x)], u), \quad i = 1, \dots, n, \tag{1}$$

where $\varphi_1, \dots, \varphi_n$ are unknown functions belonging to a certain function class G, which is defined below.

In general, the solutions of a functional equations in a single variable depends on an arbitrary functions (cf. [3], p. 44—45) and conditions ensuring the uniqueness of a solution are of a particular importance (see also [4]). In the present paper we consider this problem.

For the equation

$$\varphi(x) = h(x, \varphi[f_1(x)], \dots, \varphi[f_n(x)], u)$$

the corresponding problem has been investigated in my preceeding paper [2].

2. Let us introduce the notation:

$$a_{ik}^1 = \begin{cases} a_{ik}, & i \neq k, \\ 1 - a_{ik}, & i = k, \end{cases} \quad i, k = 1, \dots, n, \tag{2}$$

$$a_{ik}^{r+1} = \begin{cases} a_{1,1}^r a_{i+1,k+1}^r + a_{i+1,1}^r a_{1,k+1}^r, & i \neq k, \\ a_{1,1}^r a_{i+1,k+1}^r - a_{i+1,1}^r a_{1,k+1}^r, & i = k, \end{cases} \tag{3}$$

$$r = 1, \dots, n-1, \; i, k = 1, \dots, n-r.$$

We assume

$$a_{ii}^r > 0, \; r = 1, \dots, n, \; i = 1, \dots, n+1-r. \tag{4}$$

We shall use the followinц fixed-point theorem for n transformations (cf. [5]).

Theorem 1. *Let* (X_i, d_i), $i=1, \ldots, n$ *be complete metric spaces. Suppose that the transformations* $T_i\colon X_1\times\ldots\times X_n\to X_i$, $i=1, \ldots, n$ *fulfil the following conditions*

$$(5)\qquad d_i[T_i(x_1, \ldots, x_n), T_i(y_1, \ldots, y_n)] \leqq \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k),$$

$$x_k, y_k\in X_k, \quad i, k = 1, \ldots, n,$$

where $a_{ik}>0$. *If the numbers* a_{ik}^r, $r=1, \ldots, n$, $i, k=1, \ldots, n+1-r$, *defined by* (2) *and* (3) *fulfil the inequalities* (4), *then the system of equations*

$$x_i = T_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n,$$

has exactly one solution $x_i\in X_i$, $i=1, \ldots, n$. *Moreover*

$$x_i = \lim_{k\to\infty} x_{ik}, \quad i = 1, \ldots, n,$$

where $x_{i,0}\in X_i$, $i = 1, \ldots, n$, *are arbitrarily chosen and*

$$x_{i,k+1} = T_i(x_{1,k}, \ldots, x_{n,k}), \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots .$$

3. Now we are going to establish a theorem on the existence and uniqueness of solution of the system of equations (1) in special class functions.

We assume

Hypothesis 1.

(i) Let $(B, \|\cdot\|)$ be a Banach space. The functions $h_i\colon I\times B^n\times R\to B$, $I=\langle 0, \infty)$, $R=(-\infty, +\infty)$ are continuous in $I\times B^n\times R$, $i=1, \ldots, n$.

(ii) There exist functions $L_{ik}\colon I\to(0, \infty)$, $i, k=1, \ldots, n$ such that for every $x\in I$, $u\in R$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)\in B^n$ we have

$$\|h_i(x, y_1, \ldots, y_n, u)-h_i(x, z_1, \ldots, z_n, u)\| \leqq \sum_{k=1}^{n} L_{ik}(x)\|y_k-z_k\|.$$

(iii) The functions $f_{ik}\colon I\to I$ are continuous in I.

(iv) Let the function $L\colon I\to R$ be locally bounded. For every $u\in R$ there exist constants N_i and points $(r_{1,i}, \ldots, r_{n,i})\in B^n$, $i=1, \ldots, n$ such that for $x\in I$

$$J_i = \|h_i(x, r_{1,i}, \ldots, r_{n,i}, u)\| \leqq N_i \exp\big(L(x)\big).$$

(v) There exist constants a_{ik} such that

$$L_{ik}(x) \exp\big(L[f_{ik}(x)]\big) \leqq a_{ik} \exp[L(x)], \quad i = 1, \ldots, n, \quad x\in I.$$

We define G as the space of those functions $\varphi\colon I\to B$, which are continuous in I and fulfilled condition

$$(6)\qquad \|\varphi(x)\| = O\big(\exp(L(x))\big)^{1}).$$

[1]) $\|\varphi(x)\|=O(\exp(L(x)))$ denotes that there exists constant $M\geqq 0$ such that $\|\varphi(x)\|\leqq M \exp(L(x))$, $x\in I$.

For $\varphi\in G$ we define the norm (cf. also [1])

$$|\varphi| = \sup_{x\in I}\left(\|\varphi(x)\|\exp(-L(x))\right).$$

We can verify that $(G, |\cdot|)$ is a Banach space.

Remark. If $\varphi_m\in G$ converges to $\varphi\in G$ in the sense of norm $|\cdot|$, then also φ_m converges to φ uniformly in $\langle 0, d\rangle$, for every $d>0$.

Now we shall prove the following

Theorem 2. *Suppose that Hypothesis* 1 *is fulfilled. If the numbers* a_{ik}^r, $r=1, \dots, n$, $i, k=1, \dots, n+1-r$, *defined by* (2) *and* (3) *fulfil the inequalities* (4), *then the system of equations* (1) *has exactly one solution* $\varphi_i\in G$, $i=1, \dots, n$, *given as the limit of successive approximations.*

PROOF. Let $u\in R$ be fixed. We define the transform

$$T_i\colon X_1\times\dots\times X_n \to X_i,\ X_k = G,\ \Phi_i = T_i(\varphi_1, \dots, \varphi_n),\ \varphi_k\in G,\ i, k = 1, \dots, n$$

such that

$$\Phi_i(x) = h_i\left(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_n[f_{i,n}(x)], u\right), \quad i = 1, \dots, n. \tag{7}$$

We shall prove that if $\varphi_i\in G$, $i=1, \dots, n$, then $\Phi_i\in G$, $i=1, \dots, n$. From Hypothesis 1 we see that Φ_i, $i=1, \dots, n$ are continuous in I and $\Phi_i(x)\in B$ for $x\in I$, $i=1, \dots, n$. Also, by (ii)—(v), we have

$$\begin{aligned}
\|\Phi_i(x)\| &\leqq \|h_i(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_n[f_{i,n}(x)], u)-h_i(x, r_{1,i}, \dots, r_{n,i}, u)\|+J_i \leqq \\
&\leqq \sum_{k=1}^{n} L_{ik}(x)\|\varphi_k[f_{ik}(x)]-r_{k,i}\|+J_i \leqq \\
&\leqq \sum_{k=1}^{n} L_{ik}(x)|\varphi_k-r_{k,i}|\exp\left(L[f_{ik}(x)]\right)+J_i \leqq \\
&\leqq \sum_{k=1}^{n} a_{ik}|\varphi_k-r_{k,i}|\exp\left(L(x)\right)+J_i \leqq \\
&\leqq \exp\left(L(x)\right)\left[\sum_{k=1}^{n} a_{ik}|\varphi_k-r_{k,i}|+N_i\right].
\end{aligned}$$

So $\Phi_i\in G$, $i=1, \dots, n$. Let now φ_i, $\psi_i\in G$, $i=1, \dots, n$ and

$$\Phi_i = T_i(\varphi_1, \dots, \varphi_n), \quad \Psi_i = T_i(\psi_1, \dots, \psi_n).$$

From (ii) and (v) we obtain the inequalities

$$\|\Phi_i(x)-\Psi_i(x)\| \leqq$$

$$\leqq \|h_i(x, \varphi[f_{i,1}(x)], \ldots, \varphi_n[f_{i,n}(x)], u)-h_i(x, \psi_1[f_{i,1}(x)], \ldots, \psi_n[f_{i,n}(x)], u)\| \leqq$$

$$\leqq \sum_{k=1}^{n} L_{ik}(x)\|\varphi_k[f_{ik}(x)]-\psi_k[f_{ik}(x)]\| \leqq$$

$$\leqq \sum_{k=1}^{n} L_{ik}(x)|\varphi_k-\psi_k| \exp\big(L[f_{ik}(x)]\big) \leqq$$

$$\leqq \exp\big(L(x)\big) \sum_{k=1}^{n} a_{ik}|\varphi_k-\psi_k|.$$

Hence

$$|\Phi_i-\Psi_i| \leqq \sum_{k=1}^{n} a_{ik}|\varphi_k-\psi_k|, \quad i = 1, \ldots, n,$$

and relation (5) is fulfilled. In view of (4) and Theorem 1, we obtain Theorem 2, which completes the proof.

4. Now we shall consider the problem of the continuous dependence of solutions of the system of equations (1) on parameter u.

We assume

HYPOTHESIS 2.

There exists constants M_i and functions $A_i\colon I\to I$ such that for every $x\in I$, $u_1, u_2\in R$, $(z_1, \ldots, z_n)\in B^n$

$$\|h_i(x, z_1, \ldots, z_n, u_1)-h_i(x, z_1, \ldots, z_n, u_2)\| \leqq A_i(x)|u_1-u_2|$$

and

$$\sup_{x\in I}\big[A_i(x)\exp\big(-L(x)\big)\big] \leqq M_i, \quad i = 1, \ldots, n.$$

Now we shall prove

Theorem 3. *Suppose that hypotheses* 1 *and* 2 *are fulfilled. If, moreover,*

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} = p < 1, \tag{8}$$

then the system of equations (1) *for every* $u\in R$ *has exactly one solution* $\varphi_i\colon I\times\{u\}\to B$, $\varphi_i\in G$, $i=1, \ldots, n$. *Moreover, the functions* $\varphi_i(x, u)$, $i=1, \ldots, n$ *are continuous in* $I\times R$.

PROOF. Let $\Phi=(\varphi_1, \ldots, \varphi_n)$, $\varphi_i\in G$, $i=1, \ldots, n$. We define the transform

$$T_{i,u}(\Phi)(x) = h_i(x, \varphi_1[f_{i,1}(x)], \ldots, \varphi_n[f_{i,n}(x)], u), \quad i = i, \ldots, n.$$

Let $\Psi=(\psi_1, \ldots, \psi_n)$, $\psi_i\in G$, $i=1, \ldots, n$. As in preceding theorem we can obtain the relation

$$|T_{i,u}(\Phi)-T_{i,u}(\Psi)| \leqq \sum_{k=1}^{n} a_{ik}|\varphi_k-\psi_k|, \tag{9}$$

and next by (8)

$$\sum_{i=1}^{n} |T_{i,u}(\Phi)-T_{i,u}(\Psi)| \leqq p \sum_{k=1}^{n} |\varphi_k-\psi_k|. \tag{10}$$

Consequently, from Banach's fixed-point theorem for contraction maps, for every $u\in R$ there exists a unique fixed point $\Phi=(\varphi_1(x, u), \dots, \varphi_n(x, u))$, $\varphi_i\in G$, $i=1, \dots, n$ of transformation $T_u=(T_{1,u}, \dots, T_{n,u})$, i.e. unique solution $\varphi_i(x, u)$, $\varphi_i\in G$, $i=1, \dots, n$ of the system of equations (1).

So we have

$$T_{i,u}(\Phi(x, u)) = \varphi_i(x, u),\ T_{i,u_1}(\Phi(x, u_1)) = \varphi_i(x, u_1), \quad i = 1, \dots, n.$$

From Hypothesis 2, we have

$$|T_{i,u}(\Phi)-T_{i,u_1}(\Phi)| \leqq \tag{11}$$
$$\leqq \sup_{x\in I}\left[A_i(x)|u-u_1| \exp(-L(x))\right] \leqq M_i|u-u_1|, \quad i = 1, \dots, n.$$

Next, using (9) and (11) we obtain relation

$$\begin{aligned}|\varphi_i(x, u)-\varphi_i(x, u_1)| &\leqq |T_{i,u}(\Phi(x, u))-T_{i,u}(\Phi(x, u_1))| + \\ &+|T_{i,u}(\Phi(x, u_1))-T_{i,u_1}(\Phi(x, u_1))| \leqq \\ &\leqq \sum_{k=1}^{n} a_{ik}|\varphi_k(x, u)-\varphi_k(x, u_1)|+M_i|u-u_1|.\end{aligned}$$

Hence by (10) we have

$$\sum_{k=1}^{n} |\varphi_k(x, u)-\varphi_k(x, u_1)| \leqq p \sum_{k=1}^{n} |\varphi_k(x, u)-\varphi_k(x, u_1)|+|u-u_1| \sum_{k=1}^{n} M_k,$$

and consequently

$$\sum_{k=1}^{n} |\varphi_k(x, u)-\varphi_k(x, u_1)| \leqq (1-p)^{-1}|u-u_1| \sum_{k=1}^{n} M_k.$$

So we see that the functions φ_i, $i=1, \dots, n$ are continuous with respect to the variable u in R uniformly with respect to the variable x. Since φ_i, $i=1, \dots, n$ are continuous with respect to the variable x in I for fixed $u\in R$, we easily seen that φ_i, $i=1, \dots, n$ are continuous in $I\times R$, which completes the proof.

References

[1] A. Bielecki, Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires, *Bulletin de l'Academie Polonaisse des Sciences* **4** (1956), 261—264.

[2] S. Czerwik, Special solutions of a functional equation, *Ann. Polon. Math.* **31** (1975), 141—144.

[3] M. Kuczma, Functional equations in a single variable, Monografie Mat. 46, *Warszawa*, (1968).

[4] M. Kuczma, Problems of uniqueness in the theory of functional equations in a single variable, *Zeszyty Naukowe Uniwersytetu vagiellonskiego, Prace Mat.*, **14** (1970), 41—48.

[5] J. Matkowski, Some Inequalities and a Generalization of Banach's Principle, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* **21** (1973), 323—324.

(Received April 12, 1974.)