The absolute annihilator of an abelian group modulo a subgroup

By SHALOM FEIGELSTOCK (Ramat-Gan)

A group G in this paper will always be meant to be an abelian group with addition to the group operation. H will denote a subgroup of G.

The absolute annihilator, $A(G)$, of a group G has been defined [2, problem 94] as follows: $A(G)=\{g\in G/Rg=gR=0$ for every ring R on $G\}$. We consider here the following generalization of $A(G)$:

(i) Definition: The absolute annihilator of G modulo H, $(G:H)=$ $=\{g\in G/Rg\subseteqq H$, and $gG\subseteqq H$ for every ring R on $G\}$.

The absolute annihilator of G is clearly $(G:0)$.

In (ii) we will define the notion of the right and the left annihilator of G in the obvious manner. It will be shown that both these subgroups of G coincide with $(G:H)$.

A study of $(G:H)$ will be made for the case where H is an ideal in every ring on G in (iii). It will be shown that in this circumstance, the nilstufe [1, 5] of G and H are related.

$(G:H)$ will be employed to generalize the notion of a nil group (iv).

(ii) Definition: The right absolute annihilator of G modulo H, $(G:H)_r=$ $=\{g\in G/Rg\subseteqq H$ for all rings R on $H\}$. In a similar manner we define $(G:H)_l$ the left absolute annihilator of G modulo H.

Theorem 1. $(G:H)=(G:H)_r=(G:H)_l$.

PROOF. Since $(G:H)=(G:H)_r\cap(G:H)_l$ it suffices to show that $(G:H)_r=(G:H)_l$. Let $g\in(G:H)_r$ and let R be a ring on G, $R=(G,X)$, X a multiplication on G. For $a,b\in G$, define $a\overline{X}b=bXa$. $\overline{X}$ is a multiplication on G. Put $\overline{X}=(G,\overline{X})$. Since $g\in(G:H)_r$, $gXa=a\overline{X}g\in H$ for every $a\in G$. Therefore, $g\in(G:H)_l$ and hence $(G:H)_r\subseteqq$ $\subseteqq(G:H)_l$. Similarly, $(G:H)_l\subseteqq(G:H)_r$.

(iii). *Notation:* Let $R_i=(G,X_i)$, X_i a multiplication on G_i, $1\leqq i\leqq n$. Put $R_1(R_2(R_3(\dots R_n))\dots)=\{g_1X_1(g_2X_2(g_3\dots X_{n-1}g_n))\dots)|g_i\in G,\ 1\leqq i\leqq n\}$, and $\underbrace{[G,\dots,G]}_{n\text{-times}}=$ $=\underbrace{(G:(G:(\dots(G:0))\dots)}_{n\text{-times}}$.

We make the following obvious:

Observation 1: $R_1(R_2(R_3(\dots R_n))\dots)=0$ for all n rings R_i, $1\leqq i\leqq n$, on G iff $\underbrace{[G,\dots,G]}_{n\text{-times}}=G$.

The nilstufe $\nu(G)$ of G is defined, [1, 5], to be n if there exists an associative ring R on G with $R^n \neq 0$, but $R^{n+1}=0$ for every associative ring R on G. An obvious consequence of observation 1 is

Observation 2: If $\underbrace{[G, \ldots, G]}_{n-1\text{times}}=0$, then $\nu(G) \leqq n$.

Clearly, H is an ideal in every ring on G iff $(G{:}H) \supseteqq H$. In this case we obtain an ascending chain of subgroups of G, $H \subseteqq (G{:}H) \subseteqq (G{:}\,(G{:}H)) \subseteqq (G{:}\,(G{:}\,(G{:}H))) \subseteqq \ldots$

If this chain terminates at G after a finite number of steps, then we have the following:

Theorem 2. *If* $(G{:}\,H) \supseteqq H$ *and if* $\underbrace{(G{:}\,(G{:}\,(\ldots(G{:}\,H))\ldots)}_{n\text{-times}}=G$, *then* $\nu(G) < (n+1)(\nu(H)+1)$.

PROOF. Clearly $(G{:}\,(G{:}\,(\ldots(G{:}H))\ldots)=G$ implies that $R^{n+1} \subseteqq H$ for every ring R on G. Since H is an ideal in every ring on H, the restriction of every multiplication on G to H is a multiplication on H. Therefore $R^{(n+1)(\nu(H)+1)} \subseteqq H^{\nu(H)+1}=0$.

Example: Let $G=H \oplus K$, H a nil torsion group, K a torsion free group. It is easy to show that $\underbrace{(G{:}\,(G{:}\,(\ldots(G{:}\,H))\ldots)}_{\nu(K)\text{ times}}=G$.

Clearly, $(G{:}H) \supseteqq H$. Therefore, by theorem 2, $\nu(G)<2(\nu(K)+1)$. This result was obtained in [1, theorem 1].

We wish to show that the example just considered is the most general example of a mixed group that can be considered if we require that $\nu(G)<\infty$.

Theorem 3. *Let G be a mixed group. If $\nu(G)<\infty$ then G splits, and the torsion part of G is nil and coincides with the maximal divisible subgroup of G.*

PROOF. Let $G=D \oplus K$, D the maximal divisible subgroup of G. If D is not a torsion group then $\nu(D)=\infty$ and hence $\nu(G)=\infty$. Therefore D is a torsion group and hence nil [2, 4]. If K is not torsion free then K has a finite cyclic direct summand [3, theorem 9], in which case $\nu(K)=\infty$, and hence $\nu(G)=\infty$. Therefore K is torsion free, and D is the torsion part of G.

Theorem 4. *Let $G=D \oplus K$, D the maximal divisible subgroup of G, and D nil. If $(G{:}K) \supseteqq K$, then $\nu(G)=\nu(K)$.*

PROOF. The following are easily shown to be true for every multiplication on G:

a) $D^2=0$
b) $DK \subseteqq D$
c) $KD \subseteqq D$
d) $K^{\nu(K)+1} \subseteqq D$.

However since $(G{:}K) \supseteqq K$ we have that. $DK \subseteqq K$ and $KD \subseteqq K$. This together with b) and c) yields:

b′) $DK=0$, and
c′) $KD=0$.

$(G:K)\supseteqq K$ further implies that $K^n\subseteqq K$ for every positive integer n, which together with d) yields

d') $$K^{\nu(K)+1} = 0$$

Let $g_i\in G$, $1\leqq i\leqq \nu(K)+1$. $\prod_{i=1}^{\nu(K)+1} g_i$ is a finite sum of monomials

$$\prod_{i=1}^{\nu(K)+1} y_i,\ y\in D \text{ or } y_i\in K,\ 1\leqq i\leqq \nu(K)+1.$$

b'), c'), and d') insure that

$$\prod_{i=1}^{\nu(K)+1} y_i = 0.$$

(iv) Definition: G is said to be a nil group modulo H if $(G:H)=G$.

A nil group in the ordinary sense is a nil group modulo 0.

Example: Let $G=H\oplus K$, H a nil torsion free group, K a torsion group. It is easy to show that G is nil modulo K.

Theorem 5. *Let G be nil modulo H. If H is divisible then G/H is nil.*

PROOF. Let $\Theta: G\to G/H$ be the canonical homomorphism. The sequence $0\to H\to G\overset{\Theta}{\to} G/H\to 0$ is exact, hence the sequence

$$\mathrm{Hom}\,(G\otimes G, G)\overset{\Theta^*}{\to}\mathrm{Hom}\,(G\otimes G, G/H)\to\mathrm{Ext}\,(G\otimes G, H)=0$$

is exact. The last equality holds true since H is divisible. $\Theta^*(\mu)=\Theta\mu$ for all $\mu\in\mathrm{Hom}\,(G\otimes G, G)$. We can identify an arbitrary multiplication on G/H with an element $\bar\mu\in\mathrm{Hom}\,(G/H\otimes G/H, G/H)$ [2, theorem 118.1]. Let $g_1, g_2\in G$, and let $\bar g_1, \bar g_2$ be their respective cosets in G/H.

$$\bar\mu(\bar g_1\otimes\bar g_2)=\bar\mu\cdot(\Theta\otimes\Theta)(g_1\otimes g_2).$$

$\bar\mu\cdot(\Theta\otimes\Theta)\in\mathrm{Hom}\,(G\otimes G, G/K)$, and Θ^* is an epimorphism. Therefore there exists a $\mu\in\mathrm{Hom}\,(G\otimes G, G)$ such that $\bar\mu\cdot(\Theta\otimes\Theta)=\Theta^*(\mu)=\Theta\cdot\mu$. Therefore, $\bar\mu(\bar g_1\otimes\bar g_2)=$ $=\Theta\mu(g_1\otimes g_2)$. G is nil modulo H, which implies that $\mu(g_1\otimes g_2)\in H$, and hence $\bar\mu(\bar g_1\otimes\bar g_2)=\Theta(\mu(g_1\otimes g_2))=0$, or G/H is nil.

Theorem 6. *If $(G:H)\supseteqq H$ and if G/H is a nil group, then G is nil modulo H.*

PROOF. Since $(G:H)\supseteqq H$, H is an ideal in every ring on G, and hence every multiplication X_G on G induces a multiplication $X_{G/H}$ on G/H in a natural way. Let $g_1, g_2\in G$, and let $\bar g_1, \bar g_2$ be their respective cosets in G/H. $\bar g_1 X_{G/H}\bar g_2=\overline{g_1 X_G g_2}$. However G/H is a nil group, so that $\bar g_1 X_{G/H}\bar g_2=0$, which implies that $g_1 X_G g_2\in H$, ro that G is nil modulo H.

(v) The results obtained here suggest the following problems:

Problem 1: Characterize $(G:H)$ for H a special type of subgroup of G. For example, if G is a torsion group, and if G_p is the p-primary component of G, p a prime, then

$$(G: G_p) = G_p \otimes \sum_{\substack{q \text{ aprime} \\ q \neq p}} G_q^1, \text{ where } G_q^1 \text{ denotes the first Ulm subgroup of } G_q.$$

Problem 2: Let K be a reduced torsion free group. Let $NS(K) = \{\nu(G)/G = D \oplus K$, D a nil torsion group$\}$. By the example following theorem 2 we have that if $n \in NS(K)$ then $\nu(K) \leqq n < 2(\nu(K)+1)$. Is the converse true?

Problem 3: Find conditions for G to be nil modulo H, H a special type of subgroup of G.

Problem 4: If G is nil modulo H, must G/H be a nil group?

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BAR ILAN UNIVERSITY
RAMAT GAN, ISRAEL