T-ideals based on multilinear identities

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This note shows that the T-ideal based on $f(x_1, \ldots, x_n)$ is also based on multilinear elements when the characteristic of the field is zero or greater than $\max_{i=1,\ldots,n} \{x_i \text{ degree of } f\}$. Furthermore this is essentially best possible.

Let K be a field. The free associative K-algebra with indeterminates $\{x_i\}_{i=1}^{\infty}$ and without constant term will be denoted by $K[X]$. The set of monomials of $K[X]$ forms a basis for $K[X]$ as a vector space over K and hence gives a canonical representation of the polynomials in $K[X]$. Define the x_i degree of a monomial to be the number of times x_i occurs in that monomial, and define the x_i degree (respectively, order) of f in $K[X]$ to be $\max \{x_i \text{ degree of } m: m \text{ is a monomial of } f\}$ (respectively, min). The degree of a monomial m is $\sum_i (x_i \text{ degree of } m)$. The degree (respectively order) of a polynomial f is $\max \{\text{degree of } m: m \text{ is a monomial of } f\}$ (respectively min). A polynomial f is linear in x_i if x_i occurs exactly once in each monomial, and f is multilinear if it is linear in each of its variables.

A T-ideal of $K[X]$ is any ideal which is mapped into itself by the endomorphisms of $K[X]$. If S is a subset of $K[X]$ then $T\langle S\rangle$ is the smallest T-ideal containing S. The elements of $T\langle S\rangle$ are exactly

$$\sum_i \alpha_i S_i(f_{i1}, \ldots, f_{in(i)})\beta_i \text{ where } S_i \in S,$$

$f_{ij} \in K[X]$ and $\alpha_i, \beta_i \in K[X] \cup K$. If I is a T-ideal and $I = T\langle S\rangle$ then I is said to be based on S. An important tool (e.g. [2]) in the study of rings satisfying a polynomial identity f, is the fact that if the characteristic of K is zero, then $T\langle f\rangle = T\langle M\rangle$ where M is a subset of the multilinear elements of $T\langle f\rangle$. The first theorem shows that the characteristic of K need only be greater than $\max_i \{x_i \text{ degree of } f\}$.

Theorem 1. *Let $f \in K[X]$. If the characteristic of K is zero or greater than $\max_i \{x_i \text{ degree of } f\}$ then $T\langle f\rangle = T\langle M\rangle$ where M is a finite set of multilinear elements in $T\langle f\rangle$.*

The proof will follow from a lemma.

Lemma 2. *Let $f(x_1, \ldots, x_n) \in K[X]$, and write $f = f_0 = h_0 + g_0$ where each monomial of h_0 has x_1 degree d with d the x_1 degree of f, and g_0 has x_1 degree less than d. Then* *there exists $h_{d-1}(x_1, y_1, \ldots, y_{d-1}, x_2, \ldots, x_n)$ contained in $T\langle f\rangle$, linear in $x_1, y_1, \ldots, y_{d-1}$ such that $h_{d-1}(x_1, x_1, \ldots, x_1, x_2, \ldots, x_n)=d!\,h_0(x_1, x_2, \ldots, x_n)$.*

PROOF. The proof constructs h_{d-1} by the well known [1] linearization process. Suppose $f_i=h_i+g_i$ has been defined with $f_i\in T\langle f\rangle$, each monomial of $h_i=h_i(x_1, y_1, \ldots, \ldots, y_i, x_2, \ldots, x_n)$ has x_1 degree $d-i$ and is linear in $y_1, \ldots, y_i$, the x_i degree of g_i is less than $d-i$ and $h_i(x_1, x_1, \ldots, x_1, x_2, \ldots, x_n)=\frac{d!}{(d-i)!}\,h_0(x_1, x_2, \ldots, x_n)$. Define $f_{i+1}(x_1, y_1, \ldots, y_{i+1}, x_2, \ldots, x_n)=f_i(x_1+y_{i+1}, y_1, \ldots, y_i, x_2, \ldots, x_n)-f_i(x_1, y_1, \ldots, \ldots, y_i, x_2, \ldots, x_n)-f_i(y_{i+1}, y_1, \ldots, y_i, x_2, \ldots, x_n)$. Thus $f_{i+1}\in T\langle f\rangle$ and $f_{i+1}=h_{i+1}+g_{i+1}$ where h_{i+1} is linear in $y_1, \ldots, y_{i+1}$, each monomial of h_{i+1} has x_1 degree $d-(i+1)$ and g_{i+1} has x_1 degree less than $d-(i+1)$. Furthermore the monomials of h_{i+1} are obtained from monomials of h_i by substituting exactly one y_{i+1} in one of the positions occupied by an x_1 in the monomials of h_i. Since there are $d-i$ positions in which to substitute y_{i+1}, in each monomial, $h_{i+1}(x_1, y_1, \ldots, \ldots, y_i, x_1, x_2, \ldots, x_n)=(d-i)\,h_i(x_1, y_1, \ldots, y_i, x_2, \ldots, x_n)$. Thus

$$h_{i+1}(x_1, x_1, \ldots, x_1, x_2, \ldots, x_n) = (d-i)\,h_i(x_1, x_1, \ldots, x_1, x_2, \ldots, x_n) =$$

$$= (d-i)\frac{d!}{(d-i)!}\,h_0(x_1, x_2, \ldots, x_n) = \frac{d!}{(d-(i+1))!}\,h_0(x_1, \ldots, x_n).$$

Hence finally $f_{d-1}=h_{d-1}+g_{d-1}$ with $f_{d-1}\in T\langle f\rangle$ and g_{d-1} of degree 0 in x_1. Thus by substituting 0 for x_1 we have $g_{d-1}\in T\langle f\rangle$ and $h_{d-1}\in T\langle f\rangle$ with $h_{d-1}(x_1, y_1, \ldots, y_{d-1}, x_2, \ldots, x_n)$ linear in $x_1, y_1, \ldots, y_{d-1}$ and $h_{d-1}(x_1, x_1, \ldots, x_1, x_2, \ldots, x_n)=d!\,h_0(x_1, \ldots, x_n)$.

PROOF OF THEOREM 1. As in the lemma $f=h_0+g_0$ with h_0 of degree and order d in x_1 and g_0 of degree less than d. Thus $h_{d-1}\in T\langle f\rangle$ and since $d!$ is invertible in K, $h_0\in T\langle h_{d-1}\rangle$, which in turn means $g_0\in T\langle f\rangle$. By induction $T\langle g_0\rangle=T\langle M_1\rangle$ where M_1 is a finite set of multilinear elements in $T\langle g_0\rangle\subseteq T\langle f\rangle$, and $T\langle h_{d-1}\rangle=T\langle M_2\rangle$ where M_2 is a finite set of multilinear elements in $T\langle h_{d-1}\rangle\subseteq T\langle f\rangle$. Thus $T\langle f\rangle=T\langle h_{d-1}, g_0\rangle= =T\langle M_1\cup M_2\rangle$ and the theorem is proven.

To indicate that this is best possible we prove:

Theorem 3. *$T\langle x^n\rangle=T\langle M\rangle$ where M is the set of multilinear identities in $T\langle x^n\rangle$ if and only if the characteristic of K is zero or greater than n.*

Lemma 4. *Let V_n be the set of multilinear identities of $T\langle x^n\rangle$ in the variables $x_1, \ldots, x_n$. Then*

$$V_n = \{c\cdot \sum_{\sigma\in S_n} x_{\sigma(1)}\ldots x_{\sigma(n)}\colon\ c\in K\}.$$

PROOF. It is easy to see that linearization process of lemma 2 yields $\sum_{\sigma\in S_n} x_{\sigma(1)}\ldots \ldots x_{\sigma(n)}\in V_n$. If $m\in V_n$ then $m=\sum_i \alpha_i f_i^n \beta_i$. Let $f_i=a_{i1}x_1+\ldots+a_{in}x_n+\gamma_i$ where the order of γ_i is greater than one, $a_{ij}\in K$. Thus

$$m = \sum_i c_i(a_{i1}x_1+\ldots+a_{in}x_n)^n+r$$

where $c_i, a_{ij} \in K$ and the order of r is greater than n. Since m has degree n, $r=0$. The multilinear monomials of $(a_{i1}x_1+\dots+a_{in}x_n)^n$ are thus all possible $x_{\sigma(1)}\dots x_{\sigma(n)}$ $\sigma\in S_n$ with coefficient $a_{i1}a_{i2}\dots a_{in}$. Hence the multilinear part of $\sum_i c_i(a_{i1}x_1+\dots$ $\dots+a_{in}x_n)^n$ is $\sum_i c_i(a_{i1}\dots a_{in}) \sum_{\sigma\in S_n} x_{\sigma(1)}\dots x_{\sigma(n)}$ which equals m. Thus

$$V_n = \Big\{c\cdot \sum_{\sigma\in S_n} x_{\sigma(1)}\dots x_{\sigma(n)}\colon\ c\in K\Big\}.$$

PROOF OF THEOREM 3. By theorem 2, if the characteristic of K is zero or greater than n then $T\langle x^n\rangle = T\langle M\rangle$. Suppose now that the characteristic of K is k with $0<k\leqq n$. Any element q in $T\langle M\rangle$ equals $\sum_{\sigma\in S_n} f_{\sigma(1)}\dots f_{\sigma(n)}+g$ where f_i have degree one and g has order $n+1$. Thus to have monomials of degree n and x degree n, $f_i=c_ix+f_i'$, so that $q=c_1\dots c_n \sum_{\sigma\in S_n} x^n+g'=n!c_1\dots c_nx^n+g'$ where g' has no monomial of degree n which is also of x degree n. Thus for x^n to be in $T\langle M\rangle$ requires $g'=0$ and $n!c_1\dots c_n=1$. Since $0<k\leqq n$, this last condition is never satisfied.

References

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[2] D. KRAKOWSKI and A. REGEV, The polynomial identities of a Grassman algebra. *Trans. Amer. Math. Soc.* **182**, (1973).

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