On the mean convergence of interpolatory processes

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1. *Introduction and preliminary results.*

1.1. Let us denote by

$$x_{kn} \equiv \cos \vartheta_{kn} = \cos \frac{2k-1}{2n}\pi \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots) \tag{1.1}$$

the roots of the Chebysheff polynomials $T_n(x) = \cos n\vartheta$ $(x = \cos \vartheta)$, $L_n(f; x) = \sum_{k=1}^{n} f(x_{kn}) l_{kn}(x)$ the Lagrange interpolatory polynomials of degree $\leqq n-1$, $H_n(f; x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(x)$ the Hermite—Fejér interpolatory polynomials of degree $\leqq 2n-1$. Here

$$l_{kn}(x) = \frac{T_n(x)}{T_n'(x_{kn})(x-x_{kn})} = \frac{(-1)^{k+1} T_n(x) \sqrt{1-x_{kn}^2}}{n(x-x_{kn})} \quad (k = 1, 2, \ldots, n), \tag{1.2}$$

$$h_{kn}(x) = \left[1 - \frac{T_n''(x_{kn})}{T_n'(x_{kn})}(x-x_{kn})\right] l_{kn}^2(x) = \left[\frac{T_n(x)}{n(x-x_{kn})}\right]^2 (1 - x x_{kn}) \quad (k = 1, 2, \ldots, n). \tag{1.3}$$

As it is well known

$$L_n(f; x_{kn}) = f(x_{kn}) \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots), \tag{1.4}$$

$$H_n(f; x_{kn}) = f(x_{kn}), \quad H_n'(f; x_{kn}) = 0 \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots). \tag{1.5}$$

1.2. According with a classical result of L. FEJÉR [1] if $f(x) \in C[-1, 1]$ (i.e. $f(x)$ is a continuous function on $[-1, 1]$) we have with the notation $\|g(x)\| = \max_{-1 \leqq x \leqq 1} |g(x)|$

$$\lim_{n\to\infty} \|H_n(f; x) - f(x)\| = 0. \tag{1.6}$$

So

$$\lim_{n\to\infty} \int_{-1}^{1} [H_n(f; x) - f(x)]^2 \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if} \quad f \in C[-1, 1]. \tag{1.7}$$

For the L_n process as a special case of the paper P. Erdős and P. Turán shows ([2], (13))

$$\lim_{n\to\infty}\int_{-1}^{1}[L_n(f;x)-f(x)]^2\frac{dx}{\sqrt{1-x^2}}=0 \quad \text{if} \quad f\in C[-1,1]. \tag{1.8}$$

1.3. In his paper [3] P. Turán investigated the quasi Hermite—Fejér process from the point of view of uniform convergence. More exactly he considered the polynomial $H_n^*(f;x)$ of degree $\leqq 2n-2$ whose first derivative vanishes at all nodes except at the exceptional $\eta_{1(n)}=x_{r(n),n}$, i.e.

$$H_n^*(f;x_{kn})=f(x_{kn}) \quad (k=1,2,\dots,n);\quad H_n^{*\prime}(f;x_{kn})=0 \quad (x_{kn}\neq\eta_{1(n)}). \tag{1.9}$$

Later in [4] we considered the polynomial $H_n^{**}(f;x)$ of degree $\leqq 2n-3$ such that

$$H_n^{**}(f;x_{kn})=f(x_{kn}) \quad (k=1,2,\dots,n);\; H_n^{**\prime}(f;x_{kn})=0 \;(x_{kn}\neq\eta_{1(n)},\eta_{2(n)}). \tag{1.10}$$

In both papers phenomena were found which were unexpected after (1.6). (In the papers [5] and [6] were considered the processes whose first derivative vanishes at all nodes except k or $m_n(m_n\nearrow\infty)$ exceptional roots.)

1.4. In connection with the good mean convergence behaviour of the L_n and H_n processes (see. (1.7) and (1.8)) P. Turán asked whether the "middle" processes (H_n^*, H_n^{**}, a.s.o.) preserve or not the mentioned good convergence property. The aim of this paper to give some theorems in this direction.

2. *New results.*

The following two theorems show that the H_n^* and the H_n^{**} sometimes are not good processes.

Theorem 2.1. *If p is a fix positive even integer then for any $f\in C[-1,1]$*

$$\left\{\int_{-1}^{1}[H_n^*(f;x)-H_n(f;x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac1p}=\begin{cases}O\left(n^{-\frac1p}\right) & \text{if } |\eta_{1(n)}|\leqq 1-\varepsilon\ (\varepsilon>0),\\ O\left(n^{1-\frac1p}\right) & \text{if } \eta_{1(n)}=x_{1n},\end{cases} \tag{2.1}$$

or for the convergence of H_n^* process we have

(2.2)

$$\left\{\int_{-1}^{1}[H_n^*(f;x)-f(x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac1p}=\begin{cases}o(1)+O\left(n^{-\frac1p}\right) & \text{if } |\eta_{1(n)}|\leqq 1-\varepsilon \quad (\varepsilon>0),\\ O\left(n^{1-\frac1p}\right) & \text{if } \eta_{1(n)}=x_{1n}.\end{cases}$$

Further for a suitable $f_1(x)\in C[-1,1]$ we get[1])

$$\left\{\int_{-1}^{1}[H_n^*(f_1;x)-H_n(f_1;x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac1p}\sim\begin{cases}n^{-\frac1p} & \text{if } |\eta_{1(n)}|\leqq 1-\varepsilon \quad (\varepsilon>0),\\ n^{1-\frac1p} & \text{if } \eta_{1(n)}=x_{1n}\end{cases} \tag{2.3}$$

[1]) $a_n\sim b_n$ means that $a_n=O(b_n)$ and $b_n=O(a_n)$.

Theorem 2.2. *If p is a fix positive even integer then we can choose the sequences $\{\eta_{1(n)}\}$ and $\{\eta_{2(n)}\}$ such that for any $f\in C[-1,1]$*

(2.4)
$$\left\{\int_{-1}^{1}[H_n^{**}(f;x)-H_n(f;x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{p}}=\begin{cases}O(n^{-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ O(n^{1-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ O(n^{3-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\end{cases}$$

or

(2.5)
$$\left\{\int_{-1}^{1}[H_n^{**}(f;x)-f(x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{p}}=\begin{cases}o(1)+O(n^{-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ O(n^{1-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ O(n^{3-\frac{1}{p}}) & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\}.\end{cases}$$

Further for a suitable $f_2\in C[-1,1]$

(2.6)
$$\left\{\int_{-1}^{1}[H_n^{**}(f_2;x)-H_n(f_2;x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{p}}\sim\begin{cases}n^{-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ n^{1-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\},\\ n^{3-\frac{1}{p}} & \text{for certain } \{\eta_{1(n)}\},\{\eta_{2(n)}\}.\end{cases}$$

Finally we mention the following interesting result.

Theorem 2.3. *For any $f\in C[-1,1]$*

$$\text{(2.7)}\quad \int_{-1}^{1}[H_n^{*}(f;x)-H_n(f;x)]\frac{dx}{\sqrt{1-x^2}}=\int_{-1}^{1}[H_n^{**}(f;x)-H_n(f;x)]\frac{dx}{\sqrt{1-x^2}}=0$$

or

$$\text{(2.8)}\quad \int_{-1}^{1}[H_n^{*}(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}=\int_{-1}^{1}[H_n^{**}(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}=$$
$$=\int_{-1}^{1}[H_n(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}=o(1)$$

further

$$\text{(2.9)}\quad \left|\int_{-1}^{1}[L_n(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}\right|=O\left(\omega_m\left(f;\frac{1}{n}\right)\right)$$

or

$$\text{(2.10)}\quad \int_{-1}^{1}[L_n(f;x)-H_n(f;x)]\frac{dx}{\sqrt{1-x^2}}=o(1).$$

Here $\omega_m(f;t)$ is the m-th modulus of smoothness of $f(x)$.

Remark. Some theorems at $p=\infty$ were proved for the H_n^* and the H_n^{**} in [3] and [4].

3. PROOFS.

3.1. PROOF of theorem 2.1. As Turán proved in [3], (5.1)

$$H_n^*(f;x) = \sum_{k=1}^{n} f(x_{kn})h_{kn}(x) + \frac{T_n^2(x)}{x-\eta_1}\frac{1}{n^2}\sum_{k=1}^{n} x_{kn} f(x_{kn}), \tag{3.1}$$

so by $H_n(f;x) = \sum_{k=1}^{n} f(x_{kn})h_{kn}(x)$ for any positive even integer p

(3.2)

$$\left\{\int_{-1}^{1}[H_n^*(f;x)-H_n(f;x)]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{p}} = \left\{\int_{-1}^{1}\left[\frac{T_n^2(x)}{x-\eta_1}\frac{1}{n^2}\sum_{k=1}^{n} x_{kn} f(x_{kn})\right]^p\frac{dx}{\sqrt{1-x^2}}\right\}^{\frac{1}{p}} =$$

$$= \frac{1}{n^2}\left|\sum_{k=1}^{n} x_{kn} f(x_{kn})\right|\left[\int_{-1}^{1}\frac{T_n^{2p}(x)}{(x-\eta_1)^p}\frac{dx}{\sqrt{1-x^2}}\right]^{\frac{1}{p}} = W.$$

Here $\sum_{k=1}^{n} x_{kn} f(x_{kn}) = O(n)$ further using for the polynomial $Q_{2np-p} = T_n^{2p}(x)(x-\eta_1)^{-p}$ of degree $2np-p$ the Gauss—Jacobi quadrature formula we have

$$W \leqq \frac{c}{n}\left[\frac{\pi}{np}\sum_{k=1}^{np}\frac{T_n^{2p}\left(\cos\frac{2k-1}{2np}\pi\right)}{\left(\cos\frac{2k-1}{2np}\pi-\eta_1\right)^p}\right]^{\frac{1}{p}} = V.$$

Now $\left|T_n\left(\cos\frac{2k-1}{2np}\pi\right)\right| \geqq c_1 > 0$, $\cos\frac{2k-1}{2np}\pi - \cos\frac{2s-1}{2n}\pi = -2\sin\left(\frac{2k-1}{4np}+\frac{2s-1}{4n}\right)\pi\sin\left(\frac{2k-1}{4np}-\frac{2s-1}{4n}\right)\pi$ where $\eta_1 = \cos\frac{2s-1}{2n}\pi$. If $|\eta_1| \leqq 1-\varepsilon$ then $s \sim n$ on the other hand for $\eta_1 = x_{1n}$, $s=1$. So

$$V \sim \frac{1}{n}\left[\frac{1}{n}\sum_{\substack{k=1\\k\neq s}}^{n}\frac{1}{\left(\frac{k-s}{n}\right)^p\left(\frac{k+s}{n}\right)^p}\right]^{\frac{1}{p}} \sim \begin{cases} n^{-\frac{1}{p}} & \text{for } |\eta_1| \leqq 1-\varepsilon, \\ n^{1-\frac{1}{p}} & \text{for } \eta_1 = x_{1n} \end{cases}$$

according with (2.1).

To prove (2.2) let us consider that

$$H_n^*(f;x)-f(x) = H_n^*(f;x)-H_n(f;x)H_n(f;x)-f(x). \tag{3.3}$$

Using (1.7) and (2.1) we obtain (2.2).

Finally, if we take the function $f_1(x)=x$ into account we get that $\sum_{k=1}^{n} x_{kn} f_1(x_{kn}) = \sum_{k=1}^{n} x_{kn}^2 \sim n$. The remaining parts are the same as above.

3.2. PROOF of theorem 2.2. As we proved in [4], (3.10)

$$H_n^{**}(f;x)=$$

$$=\sum_{k=1}^{n} f(x_{kn})h_{kn}(x)+\frac{T_n^2(x)}{(x-\eta_1)(\eta_2-\eta_1)n^2}\left[\eta_2\sum_{j=1}^{n} f(x_{jn})x_{jn}+\sum_{j=1}^{n} f(x_{jn})(1-2x_{jn}^2)\right]+$$

$$+\frac{T_n^2(x)}{(x-\eta_2)(\eta_1-\eta_2)n^2}\left[\eta_1\sum_{j=1}^{n} f(x_{jn})x_{jn}+\sum_{j=1}^{n} f(x_{jn})(1-2x_{jn})\right].$$

To prove the relations (2.4) and (2.5) let e.g. $\eta_{1(n)}=x_{\left[\frac{n}{4}\right],n}$ $\eta_2=x_{\left[\frac{n}{2}\right],n}$ (so $\eta_1-\eta_2\geqq$ $\geqq c>0$); $\eta_1=x_{\left[\frac{n}{4}\right],n}$ $\eta_2=x_{\left[\frac{n}{4}\right]+1,n}$ $\left(\text{so } \eta_1-\eta_2\sim\frac{1}{n}\right)$; $\eta_1=x_{1n}$, $\eta_2=x_{2n}$ $\left(\text{so } \eta_1-\eta_2\sim\frac{1}{n^2}\right)$, respectively. Using that $[\ldots]=O(n)$ and the above mentioned quadrature formula, we obtain the relations.

For (2.6) we can use the same η's and the function $f_2(x)=1-2x^2$ because $\sum_{k=1}^{n} x_k\cdot$ $\cdot(1-2_{kn}^2)=0$ and $\sum_{k=1}^{n}(1-2x_{kn}^2)^2\sim n$.

The remaining part is the same as above.

3.3. PROOF of *theorem* 2.3. By $Q_{n-1}(x)=\frac{T_n(x)}{(x-\eta_1)n^2}\sum_{k=1}^{n} x_{kn}f(x_{kn})$ we obtain from (3.1)

$$\int_{-1}^{1}[H_n^*(f;x)-H_n(f;x)]\frac{dx}{\sqrt{1-x^2}}=\int_{-1}^{1}T_n(x)Q_{n-1}(x)\frac{dx}{\sqrt{1-x^2}}=0.$$

We can use similar argument for H_n^{**}. We get (2.8) by (3.3) and (2.7). To prove (2.9) let $P_n(f;x)$ be a polynomial such that $\max_{-1\leqq x\leqq 1}|f(x)-P_n(f;x)|\leqq c\omega_m\left(f;\frac{1}{n}\right)$. Then

$$\left|\int_{-1}^{1}[L_n(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}\right|=\left|\int_{-1}^{1}L_n(f;x)\frac{dx}{\sqrt{1-x^2}}-\int_{-1}^{1}P_n(f;x)\frac{dx}{\sqrt{1-x^2}}+\right.$$

$$\left.+\int_{-1}^{1}[P_n(f;x)-f(x)]\frac{dx}{\sqrt{1-x^2}}\right|\leqq\left|\frac{\pi}{n}\sum_{k=1}^{n} f(x_{kn})-\frac{\pi}{n}\sum_{k=1}^{n} P_n(f;x_{kn})\right|+c_1\omega_m\left(f;\frac{1}{n}\right)\leqq$$

$$\leqq c_2\,\omega_m\left(f;\frac{1}{n}\right)$$

Using that $L_n(f;x)-H_n(f;x)=L_n(f;x)-f(x)+f(x)-H_n(f;x)$ and (1.6) we get (2.10).

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(Received 22 May, 1974)