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# Hosszú's functional equation on the unit interval is not stable

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**Abstract.** We prove that for every $\varepsilon > 0$ there exists a function $f : (0,1) \to \mathbb{R}$ satisfying the inequality

$$|f(x+y-xy) + f(xy) - f(x) - f(y)| \leq \varepsilon \text{ for } x, y \in (0,1),$$

such that for every solution $h : (0,1) \to \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(x) - h(x)| : x \in (0,1)\} = \infty.$$

The same result holds if we replace $(0,1)$ by any interval with ends 0 and 1.

The functional equation

$$h(x+y-xy) + h(xy) = h(x) + h(y)$$

is referred to as the Hosszú's equation. It is well known (cf. [1]) that for every function $h : \mathbb{R} \to \mathbb{R}$ satisfying the Hosszú's equation there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that

$$h(x) = A(x) + C.$$

By the unit interval we understand any interval with ends 0 and 1 and by $\mathbb{R}_+$ we mean the interval $[0, \infty)$. Since $U$ is closed for operations $(x,y) \to xy$, $(x,y) \to x+y-xy$ one can study the Hosszú's equation on $U$. K. Lajkó proved in [2] that if a function $h : U \to \mathbb{R}$ satisfies the Hosszú's equation then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and a constant $C \in \mathbb{R}$ such that

$$h(x) = A(x) + C \text{ for } x \in (0,1) \tag{1}$$

($h$ may be arbitrarily chosen on $U \backslash (0,1)$).

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There arises a natural question concerning the stability of the Hosszú's equation. L. Losonczi proved in [3] that the Hosszú's equation on $\mathbb{R}$ is stable in the Hyers–Ulam sense, i.e. he obtained the following result.

**Theorem L.** *If $X$ is a Banach space and $f : \mathbb{R} \to X$ satisfies the functional inequality*

$$\|f(x+y-xy)+f(xy)-f(x)-f(y)\| \le \varepsilon \text{ for } x,y \in \mathbb{R}$$

*with an $\varepsilon \ge 0$, then there exists a unique function $h : \mathbb{R} \to X$ satisfying the Hosszú's equation such that*

$$\|f(x)-h(x)\| \le 20\varepsilon \text{ for } x \in \mathbb{R}.$$

Surprisingly, in the case where $f$ is defined on the unit interval, the answer to the question of stability is negative (cf. [4]). For every $\varepsilon > 0$ we can find a function $f_\varepsilon : U \to \mathbb{R}$ such that

$$|f_\varepsilon(x+y-xy)+f_\varepsilon(xy)-f_\varepsilon(x)-f_\varepsilon(y)| \le \varepsilon \text{ for } x,y \in U,$$

but which can not be "approximated" by any solution of the Hosszú's equation on the unit interval.

We need the following technical lemma.

**Lemma 1.** *Suppose that $F : \mathbb{R}_+ \to \mathbb{R}$ satisfies for a certain $L \in \mathbb{R}_+$ the inequalities*

$$|F(x+y)-F(x)| \le L \text{ for } x \ge y,\ x,y \in \mathbb{R}_+, \tag{2}$$

$$|F(x)| \le L \text{ for } x \in [0,6]. \tag{3}$$

*Let*

$$g(a) := \begin{cases} -\ln a & \text{for } a \in \left(0, \dfrac{1}{2}\right], \\ -\ln(1-a) & \text{for } a \in \left[\dfrac{1}{2}, 1\right), \end{cases}$$

$$f(a) := F(g(a)) \quad \text{for } a \in (0,1).$$

*Then*

$$|f(a+b-ab)+f(ab)-f(a)-f(b)| \le 4L \text{ for } a,b \in (0,1). \tag{4}$$

Proof. Let $a,b \in (0,1)$. We are going to prove that (4) is valid. At first we prove that:

**(i).** if $a \geq b \geq \frac{1}{2}$ then $g(a+b-ab) = g(a) + g(b)$,
**(ii).** if $a \geq b$, $a \geq \frac{1}{2}$ then $|g(ab) - g(b)| \leq 3$,
**(iii).** if $a \geq b$, $\frac{1}{2} \geq b$ then $|g(a+b-ab) - g(a)| \leq 3$.

**ad (i).** We have

$$g(a+b-ab) = -\ln((1-a)(1-b)) = -\ln(1-a) - \ln(1-b) = g(a) + g(b).$$

**ad (ii).** a). Suppose that $b \geq \frac{1}{2}$.
If $ab \leq \frac{1}{2}$ then $b \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$, $ab \in \left[\frac{1}{4}, \frac{1}{2}\right]$, so

$$|g(ab) - g(b)| \leq |g(ab)| + |g(b)| \leq \ln 4 + \left|\ln\left(1 - \frac{1}{\sqrt{2}}\right)\right| \leq 3.$$

If $ab \geq \frac{1}{2}$ then

$$|g(ab) - g(b)| = |\ln(1-ab) - \ln(1-b)| = \left|\ln\left(\frac{1-b+b-ab}{1-b}\right)\right|$$
$$= \left|\ln\left(1 + b\frac{1-a}{1-b}\right)\right| \leq \ln 2 \leq 3.$$

b). Suppose that $b \leq \frac{1}{2}$. Then

$$|g(ab) - g(b)| = |\ln(ab) - \ln(b)| = |\ln(a)| \leq \ln 2 \leq 3.$$

**ad (iii).** Let $x^* := 1 - x$. Then $g(x) = g(x^*)$. Making use of the equality

$$g(a+b-ab) = g((a^*b^*)^*) = g(a^*b^*)$$

and interchanging the role of $a$ and $b$ we obtain (iii) from (ii).

Now we show that

$$|F(x) - F(y)| \leq 2L \text{ for } x, y \in \mathbb{R}_+, |x-y| \leq 3. \tag{5}$$

If $x, y \in [0, 6]$ then by (3) relation (5) is obvious. In the other case we may assume that $x \geq y$, $x \geq 6$. Then $x - y \leq 3 \leq y$ and due to (2) we have

$$|F(x) - F(y)| = |F(y + (x-y)) - F(y)| \leq L \leq 2L.$$

We are going to prove that (4) holds. Without loss of generality we may assume that $a \geq b$. Suppose that $a \geq b \geq \frac{1}{2}$. Then by (i), (ii), (5) and (2)

$$\begin{aligned}
&|f(a+b-ab) + f(ab) - f(a) - f(b)| \\
&\quad = |F(g(a+b-ab)) + F(g(ab)) - F(g(a)) - F(g(b))| \\
&\quad \leq |F(g(a+b-ab)) - F(g(a) + g(b))| \\
&\qquad + |F(g(a) + g(b)) - F(g(a))| + |F(g(ab) - F(g(b))| \leq 4L.
\end{aligned}$$

Suppose that $a \geq \frac{1}{2} \geq b$. Then due to (ii), (iii) and (5)

$$\begin{aligned}
&|f(a+b-ab)+f(ab)-f(a)-f(b)| \\
&\qquad = |F(g(a+b-ab))+F(g(ab))-F(g(a))-F(g(b))| \\
&\qquad \leq |F(g(a+b-ab))-F(g(a))|+|F(g(ab))-F(g(b))| \leq 4L.
\end{aligned}$$

Suppose that $\frac{1}{2} \geq a \geq b$. Then $b^* \geq a^* \geq \frac{1}{2}$ and we have

$$\begin{aligned}
&|f(a+b-ab)+f(ab)-f(a)-f(b)| \\
&\qquad = |f(a^*b^*)+f(a^*+b^*-a^*b^*)-f(a^*)-f(b^*)| \leq 4L. \quad \square
\end{aligned}$$

Now we are able to prove the main theorem.

**Theorem 1.** *Let $U$ be the unit interval. For every $\varepsilon > 0$ there exists a function $f : U \to \mathbb{R}$ satisfying the inequality*

$$|f(x+y-xy)+f(xy)-f(x)-f(y)| \leq \varepsilon \text{ for } x, y \in U,$$

*such that for every solution $h : U \to \mathbb{R}$ of the Hosszú's functional equation*

$$\sup\{|f(x)-h(x)| : x \in U\} = \infty.$$

*Moreover, for every $\varepsilon > 0$ and every $K > 0$ there exists a continuous bounded function $f : U \to \mathbb{R}$ which satisfies the inequality*

$$|f(x+y-xy)+f(xy)-f(x)-f(y)| \leq \varepsilon \text{ for } x, y \in U,$$

*but such that for every solution $h : U \to \mathbb{R}$ of the Hosszú's functional equation*

$$\sup\{|f(x)-h(x)| : x \in U\} \geq K\varepsilon.$$

Proof. Without loss of generality we may assume that $\varepsilon = 1$. To prove the first part of the theorem let

$$F(x) := \frac{1}{8}\ln(1+x) \quad \text{for } x \in \mathbb{R}_+.$$

Then $F$ satisfies (2) and (3) with $L = \frac{1}{4}$. Let

$$f(x) := \begin{cases} F(g(a)) & \text{for } a \in (0,1), \\ 0 & \text{for } a \in U \backslash (0,1), \end{cases}$$

where $g$ is the function defined in Lemma 1. We show that

$$|f(a+b-ab)+f(ab)-f(a)-f(b)| \leq 1 \text{ for } a, b \in U.$$

If $a = 0$ or $b = 0$ then this is obvious. For $a, b \in (0, 1)$ the relation holds by Lemma 1. As it is well known (cf. [1], p. 277), an additive function $A : \mathbb{R} \to \mathbb{R}$ is either continuous or has a dense graph in $\mathbb{R} \times \mathbb{R}$. Since $f$ is continuous on $(0, 1)$ and

$$\lim_{a \to 0} f(a) = \lim_{a \to 1} f(a) = +\infty,$$

this implies that

$$\sup\{|f(a) - A(a) - C| : a \in (0, 1)\} = \infty. \tag{6}$$

for every additive function $A$ and every constant $C$. This means that for every solution $h : U \to \mathbb{R}$ of the Hosszú's functional equation

$$\sup\{|f(a) - h(a)| : a \in U\} = \infty.$$

Now we prove the second part of the theorem. Let

$$F_n(x) := \begin{cases} \frac{1}{8}\ln(1 + x) & \text{for } x \in [0, n], \\ \frac{1}{8}\ln(1 + n) & \text{for } x \in (n, \infty). \end{cases}$$

and let

$$f_n(a) := \begin{cases} F_n(g(a)) & \text{for } a \in (0, 1), \\ F_n(n) & \text{for } a \in U \setminus (0, 1). \end{cases}$$

One can easily notice that $f_n$ is continuous and by Lemma 1

$$|f_n(a + b - ab) + f_n(ab) - f_n(a) - f_n(b)| \leq 1 \text{ for } a, b \in U.$$

We claim that for every $K > 0$ there is an $n \in \mathbb{N}$ such that

$$\sup\{|f_n(a) - h(a)| : a \in U\} \geq K$$

for all solutions $h$ of Hosszú's equation.

Otherwise, for all $n \in \mathbb{N}$ there were solutions $h_n$ of Hosszú's equation such that

$$\sup\{|f_n(a) - h_n(a)| : a \in U\} < K. \tag{7}$$

Since $h_n$ satisfies the Hosszú's equation, by (1) it has the form

$$h_n(a) = A_n(a) + C_n \quad \text{for } a \in (0, 1), \tag{8}$$

where $A_n$ is an additive function and $C_n$ is a constant. As $f_n$ is continuous, by (7) and (8) we obtain that $A_n$ is bounded in the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, so there exists $L_n \in \mathbb{R}$ such that $A_n(a) = L_n a$. One can easily check that

$\{f_n(\frac{1}{3})\}$ and $\{f_n(\frac{2}{3})\}$ are bounded sequences. This and (7) implies that $\{C_n\}$, $\{L_n\}$ are bounded sequences. Hence there exist $C, L \in \mathbb{R}$ and an increasing sequence $\{k_n\} \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} C_{k_n} = C \text{ and } \lim_{n\to\infty} L_{k_n} = L.$$

Since

$$\lim_{n\to\infty} f_n(a) = f(a) \quad \text{for } a \in (0,1),$$

we obtain that

$$|f(a) - La - C| \leq K \quad \text{for } a \in (0,1),$$

which contradicts (6). □

## References

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