The multipliers of $L_1([0, 1])$ with order convolution

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1. Introduction. Let $M([0, 1])$ denote the Banach space with the total variation norm of all bounded regular complex-valued Borel measures on the closed unit interval $[0, 1]$. If the interval $[0, 1]$ is considered as a topological semigroup under the multiplication defined by $x \circ y = \max(x, y)$, $0 \leqq x, y \leqq 1$, then a product can be introduced into $M([0, 1]$ in the following manner: if $\mu, \nu \in M([0, 1])$, then $\mu \circ \nu \in M([0, 1])$ is defined by the equations

$$\int_0^1 f(z)\, d(\mu \circ \nu)(z) = \int_0^1 \Big[\int_0^1 f(x \circ y)\, d\mu(x)\Big]\, d\nu(y) \quad (f \in C([0, 1]))$$

Of course, $C([0, 1])$ denotes the Banach space of continuous complex-valued functions on $[0, 1]$ with the usual supremum norm $\|\cdot\|_\infty$. This product operation in $M([0, 1])$ is usually called *order convolution*, and with order convolution as multiplication the space $M([0, 1])$ is a semisimple commutative Banach algebra with identity. The Banach subspace $L_1([0, 1])$ of $M([0, 1])$ consisting of the equivalence classes of Lebesgue integrable functions on $[0, 1]$ is a subalgebra of $M([0, 1])$ with respect to order convolution, and hence it is itself a commutative Banach algebra. Our purpose in this paper is to study the multipliers of $L_1([0, 1])$ with order convolution.

We recall that a mapping $T: L_1([0, 1]) \to L_1([0, 1])$ is a *multiplier* of $L_1([0, 1])$ if $T(f \circ g) = (Tf) \circ g = f \circ (Tg)$ for every f and g in $L_1([0, 1])$. Every multiplier is a bounded linear transformation since $L_1([0, 1])$ with order convolution is semisimple. In contrast to the situation for the multipliers of the group algebra $L_1(G)$ of a locally compact Abelian topological group G, every measure in $M([0, 1])$ does not define a multiplier of $L_1([0, 1])$ by means of order convolution. This is the case because $L_1([0, 1])$ is not an ideal in $M([0, 1])$ with respect to order convolution. However, in the next section we shall see that the multipliers of $L_1([0, 1])$ with order convolution correspond precisely to measures μ of the form $\mu = \alpha\delta + h$ where α is a complex number, δ is the measure with unit mass concentrated at $x = 0$, and h is in $L_1([0, 1])$. In succeeding sections we shall obtain precise descriptions of the positive and isometric multipliers of $L_1([0, 1])$. Before discussing these results it will be useful to mention some additional facts about $M([0, 1])$ and $L_1([0, 1])$ with order convolution and about multipliers, and to set some notation.

The order convolution of two elements f and g in $L_1([0, 1])$ has a relatively simple form, namely, it is determined almost every-where by the formula

$$f \circ g(x) = f(x) \int_0^x g(y)\,dy + g(x) \int_0^x f(y)\,dy.$$

Using this formula it is not difficult to see that the maximal ideal space of $L_1([0, 1])$ with order convolution is homeomorphic to the half-open interval $(0, 1]$, and that the Gel'fand transform $\hat{f}$ of f in $L_1([0, 1])$ is defined by

$$\hat{f}(x) = \int_0^x f(y)\,dy \quad (0 < x \leqq 1),$$

that is, $\hat{f}$ is the indefinite integral of f on $[0, 1]$. The algebra $L_1([0, 1])$ is without identity, but it does possess approximate identities. One such approximate identity is the sequence $\{u_n\}$ defined by $u_n(x)=n$, $0 \leqq x \leqq 1/n$, $u_n(x)=0$, $1/n < x \leqq 1$, $n=1, 2, 3, \ldots$. The identity in $M([0, 1])$ is the measure δ with unit mass concentrated at $x=0$.

If T is a multiplier of $L_1([0, 1])$ with order convolution, then there exists a unique bounded continuous function φ defined on the maximal ideal space of $L_1([0, 1])$, that is, a unique φ in $C\big((0, 1]\big)$ such that $(Tf)\hat{} = \varphi\hat{f}$, $f \in L_1([0, 1])$. Furthermore, $\|\varphi\|_\infty \leqq \|T\|$. Conversely, if $\varphi \in C\big((0, 1]\big)$ is such that for each $f \in L_1([0, 1])$ there exists some $g \in L_1([0, 1])$ for which $\hat{g} = \varphi\hat{f}$, then the equation $(Tf)\hat{} = \varphi\hat{f}$, $f \in L_1([0, 1])$, determines a multiplier of $L_1([0, 1])$. This correspondence allows us to think of a multiplier either as the mapping T or as the function φ, and we shall make frequent use of this observation.

The results concerning $M([0, 1])$ and $L_1([0, 1])$ with order convolution that we have mentioned in the preceding paragraphs and that we shall use in the succeeding sections can be found in [1, 2], whereas a general discussion of multipliers is available in [3, Chapters 0 and 1].

The Banach space of continuous complex-valued functions that vanish at infinity on the locally compact Hausdorff topological space $(0, 1]$ will be denoted by $C_0\big((0, 1]\big)$. This space can obviously be identified with the subspace of $C([0, 1])$ consisting of the functions that vanish at $x=0$. The subalgebra of $C([0, 1])$ consisting of the absolutely continuous functions will be denoted by $AC([0, 1])$, the Banach space of essentially bounded measurable functions on $(0, 1]$ by $L_\infty\big((0, 1]\big)$, and the complex numbers by $\mathbf{C}$. The symbol $\#$ will be used to indicate the end of a proof. Basic results about absolutely continuous functions that we shall use in the following sections can be found in [5, pp. 104—107].

2. The main multiplier theorem. Keeping in mind the relationship between a multiplier T and the corresponding function φ, it is easily seen that if $\varphi \in AC([0, 1])$, then φ determines a multiplier of $L_1([0, 1])$ with order convolution. Indeed, since the product of two absolutely continuous functions is absolutely continuous, it is apparent that $\varphi\hat{f} \in AC([0, 1])$ and $(\varphi)\hat{f}(0)=0$ for each $f \in L_1([0, 1])$, whence we deduce that there exists some $g \in L_1([0, 1])$ such that $\hat{g} = \varphi\hat{f}$. Clearly g is almost everywhere equal to the derivative of $\varphi\hat{f}$, that is, $g=(\varphi\hat{f})'$. Thus every function $\varphi \in AC([0, 1])$ defines a multiplier T of $L_1([0, 1])$ and $Tf=(\varphi\hat{f})'$, $f \in L_1([0, 1])$. The next theorem shows that every multiplier of $L_1([0, 1])$ can be so realized.

Theorem 1. *If $T: L_1([0, 1]) \to L_1([0, 1])$, then the following are equivalent:*

(i) *The mapping T is a multiplier of $L_1([0, 1])$ with order convolution.*

(ii) *There exists a unique $\mu \in M([0, 1])$ of the form $\mu = \alpha\delta + h$, $\alpha \in \mathbf{C}$ and $h \in L_1([0, 1])$, such that $Tf = \mu \circ f$, $f \in L_1([0, 1])$.*

(iii) *There exists a unique $\varphi \in AC([0, 1])$ such that $(Tf)\hat{} = \varphi\hat{f}$, $f \in L_1([0, 1])$.*

(iv) *There exists a unique $\varphi \in AC([0, 1])$ such that $Tf = (\varphi\hat{f})'$, $f \in L_1([0, 1])$.*

Moreover, if T is a multiplier of $L_1([0, 1])$ with order convolution, then

$$\|T\| = \|\mu\| = |\varphi(0)| + \int_0^1 |\varphi'(y)|\,dy.$$

PROOF. Evidently parts (iii) and (iv) are equivalent and the remarks preceding the statement of the theorem show that part (iii) implies part (i). Suppose there exists a unique μ in $M([0, 1])$ of the form $\mu = \alpha\delta + h$, $\alpha \in \mathbf{C}$ and $h \in L_1([0, 1])$, such that $Tf = \mu \circ f$, $f \in L_1([0, 1])$. Then given $0 < x \leqq 1$, we have for each $f \in L_1([0, 1])$

$$(Tf)\hat{}\,(x) = \hat{\mu}(x)\hat{f}(x) = [\alpha + \hat{h}(x)]\hat{f}(x).$$

Naturally $\hat{\mu}(x) = \int_0^x d\mu(y)$. Define φ on $[0, 1]$ by $\varphi(x) = \alpha + \hat{h}(x)$, $0 < x \leqq 1$, $\varphi(0) = \alpha$. Then obviously $\varphi \in AC([0, 1])$ and $(Tf)\hat{} = \varphi\hat{f}$, $f \in L_1([0, 1])$. Thus part (ii) implies part (iii).

To complete the proof of the equivalence of parts (i) through (iv) we need to show that T being a multiplier implies the existence of a unique μ of the form $\mu = \alpha\delta + h$ such that $Tf = \mu \circ f$, $f \in L_1([0, 1])$. Assume that $\varphi \in C\big((0, 1]\big)$ is such that $(Tf)\hat{} = \varphi\hat{f}$, $f \in L_1([0, 1])$.

Let $\{u_n\}$ be the approximate identity for $L_1([0, 1])$ defined in the introduction and observe that $\|Tu_n\|_1 \leqq \|T\|$, $n = 1, 2, 3, \ldots$. The symbol $\|\cdot\|_1$ of course denotes the L_1-norm. Thus $\{Tu_n\}$ is a norm bounded sequence in $M([0, 1])$, whence appealing to the Banach—Alaoglu theorem and the separability of $C([0, 1])$ [4, pp. 254 and 261] we deduce that there exists a subsequence $\{Tu_{n_k}\}$ of $\{Tu_n\}$ and a μ in $M([0, 1])$ such that

$$\lim_k \int_0^1 g(y)\,Tu_{n_k}(y)\,dy = \int_0^1 g(y)\,d\mu(y) \quad \big(g \in C\big(([0, 1]\big)\big).$$

Since T is a multiplier and $\{u_n\}$ is an approximate identity for $L_1([0, 1])$ we observe that

$$\lim_n \|Tf - T(u_n \circ f)\|_1 = \lim_n \|Tf - (Tu_n) \circ f\|_1 = 0.$$

Consequently for each g in $C_0((0, 1])$ and each $f \in L_1([0, 1])$ we have

$$\int_0^1 g(y)\,Tf(y)\,dy = \lim_k \int_0^1 g(y)\,(Tu_{n_k})\circ f(y)\,dy$$

$$= \lim_k \Big\{ \int_0^1 g(y)[Tu_{n_k}(y)\hat{f}(y) + f(y)\,(Tu_{n_k})\hat{}\,(y)]\,dy \Big\}$$

$$= \lim_k \Big\{ \int_0^1 g(y)\hat{f}(y)\,Tu_{n_k}(y)\,dy + \int_0^1 g(y)\varphi(y)f(y)\hat{u}_{n_k}(y)\,dy \Big\}$$

$$= \int_0^1 g(y)\hat{f}(y)\,d\mu(y) + \int_0^1 g(y)\varphi(y)f(y)\,dy.$$

The limiting operation on the second integral on the right hand side of the equations is established by an application of the Lebesgue dominated convergence theorem after observing that the sequence $\{\hat{u}_n\}$ converges to one pointwise on $(0, 1]$ and $\|\hat{u}_n\|_\infty = 1$ for each n.

On the other hand, $\mu \circ f \in M([0, 1])$ and straightforward calculations utilizing the definition of order convolution reveal

$$\int_0^1 g(z)\,d(\mu\circ f)(z) = \int_0^1 \Big[\int_0^1 g(x\circ y) f(x)\,dx \Big]\,d\mu(y)$$

$$= \int_0^1 g(y)\hat{f}(y)\,d\mu(y) + \int_0^1 g(y)f(y)\hat{\mu}(y)\,dy.$$

However, since $(gf)\hat{} \in C_0((0, 1])$, we see, on interchanging the order of integration twice, thar

$$\int_0^1 g(y)f(y)\hat{\mu}(y)\,dy = \int_0^1 \Big[\int_x^1 g(y)f(y)\,dy \Big]\,d\mu(x) =$$

$$= \int_0^1 \Big[\int_0^1 g(y)f(y)\,dy - \int_0^x g(y)f(y)\,dy \Big]\,d\mu(x) = (gf)\hat{}\,(1)\int_0^1 d\mu(x) - \int_0^1 (gf)\hat{}\,(x)\,d\mu(x) =$$

$$= \lim_k \Big\{ (gf)\hat{}\,(1)\int_0^1 Tu_{n_k}(x)\,dx - \int_0^1 (gf)\hat{}\,(x)\,Tu_{n_k}(x)\,dx \Big\} =$$

$$= \lim_k \Big\{ (gf)\hat{}\,(1)\int_0^1 Tu_{n_k}(x)\,dx - \int_0^1 \Big[\int_x^1 Tu_{n_k}(y)\,dy \Big]\,g(x)f(x)\,dx \Big\} =$$

$$= \lim_k \int_0^1 g(x)f(x)(Tu_{n_k})\hat{}\,(x)\,dx = \lim_k \int_0^1 g(x)f(x)\varphi(x)\hat{u}_{n_k}(x)\,dx =$$

$$= \int_0^1 g(x)f(x)\varphi(x)\,dx.$$

Hence,

$$\int_0^1 g(y)\,d(\mu\circ f)(y) = \int_0^1 g(y)\hat{f}(y)\,d\mu(y) + \int_0^1 g(y)f(y)\hat{\mu}(y)\,dy =$$

$$= \int_0^1 g(y)\hat{f}(y)\,d\mu(y) + \int_0^1 g(y)f(y)\varphi(y)\,dy = \int_0^1 g(y)\,Tf(y)\,dy.$$

Since this holds for each g in $C_0((0, 1])$, we conclude that Tf in $L_1([0, 1])$ and $\mu\circ f$ in $M([0, 1])$ define the same measure on $(0, 1]$ for each f in $L_1([0, 1])$. In particular $\mu\circ f$ on $(0, 1]$ belongs to $L_1((0, 1])$. This fact combined with the expression for $\mu\circ f$ just obtained entails that for each f in $L_1([0, 1])$ the measure $\hat{f}d\mu$ on $(0, 1]$ is absolutely continuous with respect to Lebesgue measure on $(0, 1]$.

Thus for each k there exists some $h_k\in L_1((0, 1])$ such that $\hat{u}_{n_k}d\mu=h_k$. Since the sequence $\{\hat{u}_{n_k}\}$ converges to one pointwise on $(0, 1]$ and $\|\hat{u}_{n_k}\|_\infty=1$, another application of the Lebesgue dominated convergence theorem reveals that for each $g\in L_\infty((0, 1])$ the sequence of numbers

$$\int_0^1 g(y)\hat{u}_{n_k}(y)\,d\mu(y) = \int_0^1 g(y)h_k(y)\,dy$$

is a Cauchy sequence, that is, $\{h_k\}$ is a Cauchy sequence in the weak topology on $L_1((0, 1])$. However, $L_1((0, 1])$ is weakly sequentially complete [4, p. 247] and so we see that there exists some h in $L_1((0, 1])$ such that

$$\lim_k \int_0^1 g(y)h_k(y)\,dy = \int_0^1 g(y)h(y)\,dy \quad (g\in L_\infty((0, 1])).$$

In particular, if $g\in C_0((0, 1])$, then

$$\int_0^1 g(y)h(y)\,dy = \lim_k \int_0^1 g(y)h_k(y)\,dy$$

$$= \lim_k \int_0^1 g(y)\hat{u}_{n_k}(y)\,d\mu(y) = \int_0^1 g(y)\,d\mu(y),$$

whence μ and h are seen to define the same measure on $(0, 1]$. Therefore there exists some α in $\mathbf{C}$ such that $\mu=\alpha\delta+h$ and h can obviously be considered as an element of $L_1([0, 1])$. Moreover, since δ is the identity of $M([0, 1])$ with order convolution, it is apparent that $\mu\circ f$ is in $L_1([0, 1])$ for each f in $L_1([0, 1])$ and so $Tf=\mu\circ f$, $f\in L_1([0, 1])$.

To see that μ is unique suppose that $\nu\in M([0, 1])$ is another measure such that $Tf=\nu\circ f$, $f\in L_1([0, 1])$. Then it follows easily that

$$\int_0^x d\nu(y) = \hat{\nu}(x) = \hat{\mu}(x) =$$

$$= \alpha + \int_0^x h(y)\,dy \quad (0 < x \leqq 1),$$

and from this we deduce at once that $\nu(\{0\})=\alpha=\mu(\{0\})$. The uniqueness of μ is then an immediate consequence of Theorem 4.2 in [1].

Thus part (i) implies part (ii), and the equivalence of the four parts of the theorem are established.

Obviously, if T is a multiplier of $L_1([0, 1])$ and $Tf=\mu\circ f$, $f\in L_1([0, 1])$, then $\|T\|\leqq\|\mu\|$. Moreover, the argument used in proving the implication from part (i) to part (ii) shows that μ is the weak-star limit of a sequence in $M([0, 1])$ bounded in norm by $\|T\|$, and so $\|\mu\|\leqq\|T\|$ as norm closed bounded balls in $M([0, 1])$ are weak-star closed. Consequently, since it is now evident that $\mu=\varphi(0)\delta+\varphi'$, we conclude that

$$\|T\| = \|\mu\| = |\varphi(0)|+\int_0^1 |\varphi'(y)|\,dy. \#$$

We remark that the inequality $\|\varphi\|_\infty\leqq\|T\|=\|\mu\|$ may be strict. For example, if $\varphi(x)=x-1$, $0\leqq x\leqq 1$, then $\|\varphi\|_\infty=1$ and $\|\mu\|=2$.

Two corollaries of Theorem 1 are immediate.

Corollary 1. *If T is a multiplier of $L_1([0, 1])$ with order convolution and $\varphi\in AC([0, 1])$ is such that $(Tf)\hat{} =\varphi\hat{f}$, $f\in L_1([0, 1])$, then the following are equialent:*

(i) *There exists a unique $h\in L_1([0, 1])$ such that $Tf=h\circ f$, $f\in L_1([0, 1])$.*

(ii) $\varphi(0)=0$.

It is easily seen that if T is a multiplier of $L_1([0, 1])$ and T is a compact transformation, then there exists some h in $L_1([0, 1])$ for which $Tf=h\circ f$, $f\in L_1([0, 1])$, and so $\varphi(0)=0$. We do not know whether the converse assertion is valid.

Corollary 2. *If $\mu\in M([0, 1])$, then the following are equivalent:*

(i) *There exists some $\alpha\in\mathbf{C}$ and $h\in L_1([0, 1])$ such that $\mu=\alpha\delta+h$.*

(ii) $\mu\circ L_1([0, 1])\subset L_1([0, 1])$.

This corollary is of interest since, as noted in the introduction, $L_1([0, 1])$ is not an ideal in $M([0, 1])$ with respect to order convolution.

It is also worth while noting explicitly that Theorem 1 provides an answer to an essentially classical question. Namely, for what continuous functions φ on $[0, 1]$ is it the case that for every f in $L_1([0, 1])$ the product

$$\varphi(x)\int_0^x f(y)\,dy \quad (0\leqq x\leqq 1)$$

is an indefinite integral of an element of $L_1([0, 1])$? Theorem 1 shows that φ is such a function if and only if φ is absolutely continuous.

3. Positive multipliers. A multiplier T of $L_1([0, 1])$ is said to be *positive* provided $Tf(x)\geqq 0$ almost everywhere on $[0, 1]$ whenever $f\in L_1([0, 1])$ and $f(x)\geqq 0$ almost everywhere. The next theorem gives a complete description of the positive multipliers.

Theorem 2. *If T is a multiplier of $L_1([0, 1])$ with order convolution, then the following are equivalent:*

(i) *The multiplier T is positive.*

(ii) *If $\varphi\in AC([0, 1])$ is such that $(Tf)\hat{} =\varphi\hat{f}$, $f\in L_1([0, 1])$, then $\varphi(x)\geqq 0$ for every x in $[0, 1]$ and $\varphi'(x)\geqq 0$ for almost every x in $[0, 1]$.*

(iii) *If $\mu=\alpha\delta+h$, $\alpha\in\mathbf{C}$ and $h\in L_1([0, 1])$, is such that $Tf=\mu\circ f$, $f\in L_1([0, 1])$, then $\alpha\geqq 0$ and $h(x)\geqq 0$ for almost every x in $[0, 1]$.*

PROOF. We note that for each n,

$$(Tu_n)\hat{}\,(x) = \varphi(x)\hat{u}_n(x) = n\varphi(x)x \quad \text{for} \quad 0 < x \leqq 1/n$$
$$= \varphi(x) \quad \text{for} \quad 1/n < x \leqq 1,$$

from which it follows at once that if T is positive, then $\varphi(x) \geqq 0$, $0 < x \leqq 1$. Since φ is continuous on $[0, 1]$ this also entails that $\varphi(0) \geqq 0$. Moreover, for almost every x in $[0, 1]$, if n is chosen so that $0 < 1/n < x$, then

$$\begin{aligned} Tu_n(x) &= (\varphi\hat{u}_n)'(x) \\ &= \varphi'(x)\hat{u}_n(x) + \varphi(x)u_n(x) \\ &= \varphi'(x), \end{aligned}$$

and so T positive implies $\varphi'(x) \geqq 0$ almost everywhere on $[0, 1]$. Hence part (i) implies part (ii).

Clearly part (ii) implies part (iii) since $\alpha = \varphi(0)$ and $h = \varphi'$, and part (iii) is seen to imply part (i) on observing for each $f \in L_1([0, 1])$ that

$$Tf = \mu \circ f = \alpha f + h \circ f = \alpha f + h\hat{f} + f\hat{h}. \quad \#$$

If T is a multiplier of $L_1([0, 1])$ and $(Tf)\hat{}\, = \varphi\hat{f}$, $f \in L_1([0, 1])$, then as we observed in the preceding section, it may be the case that $\|\varphi\|_\infty < \|T\|$. However, if T is positive, then this cannot happen.

Corollary 3. *If T is a positive multiplier of $L_1([0, 1])$ with order convolution and $\varphi \in AC([0, 1])$ is such that $(Tf)\hat{}\, = \varphi\hat{f}$, $f \in L_1([0, 1])$, then* $\|\varphi\|_\infty = \|T\|$.

PROOF. From Theorem 2 we see that $\varphi(x) \geqq 0$ on $[0, 1]$ and $\varphi'(x) \geqq 0$ almost everywhere on $[0, 1]$, whence $\|\varphi\|_\infty = \varphi(1)$. Moreover, by Theorem 1,

$$\begin{aligned} \|T\| &= |\varphi(0)| + \int_0^1 |\varphi'(y)|\,dy = \\ &= \varphi(0) + \int_0^1 \varphi'(y)\,dy \\ &= \varphi(1). \quad \# \end{aligned}$$

The converse of Corollary 3 may fail. Indeed, if $\varphi(x) = -x$, $0 \leqq x \leqq 1$, and T is the multiplier of $L_1([0, 1])$ determined by φ, then T is not positive and $\|T\| = \|\varphi\|_\infty = 1$.

4. Isometric multipliers. It is well known that a multiplier of the group algebra $L_1(G)$ of a locally compact Abelian topological group G is an isometry if and only if it is a constant multiple of a translation and the constant has absolute value one [6, p. 254]. In contrast, if for each x, $0 \leqq x \leqq 1$, we define the translation operator τ_x on $L_1([0, 1])$ by $\tau_x f(y) = f(x \circ y)$, then τ_x, $0 < x \leqq 1$, is not even a multiplier of $L_1([0, 1])$ with order convolution. Indeed, if $0 < x \leqq 1$ and f and g are both identically one on $[0, 1]$, then simple computations using the definition of order convolution reveal that $\tau_x(f \circ g)(y) = 2x$, $0 \leqq y \leqq x$, whereas $(\tau_x f) \circ g(y) = 2y$, $0 \leqq y \leqq x$. However, it is obvious that every multiple of τ_0 by a constant α of absolute value one, that is, $Tf=\alpha f$, $f\in L_1([0,1])$, is an isometric multiplier of $L_1([0,1])$. Theorem 3 shows that these are the only isometric multipliers. Our proof of the theorem requires a number of preliminary lemmas.

Lemma 1. *Let T be an isometric multiplier of $L_1([0,1])$ with order convolution and let $\mu\in M([0,1])$ be such that $Tf=\mu\circ f$, $f\in L_1([0,1])$. If $f\in L_1([0,1])$, then $|\mu\circ f(x)|=|\mu|\circ|f|(x)$ for almost every x in $[0,1]$.*

Proof. From Theorem 1 we know that $\mu=\alpha\delta+h$ for some $\alpha\in\mathbf{C}$ and some $h\in L_1([0,1])$. If $f\in L_1([0,1])$, then for almost every x in $[0,1]$ we have

$$|\mu\circ f(x)| = |(\alpha\delta+h)\circ f(x)|$$

$$= |\alpha f(x)+h(x)\hat{f}(x)+f(x)\hat{h}(x)| \leqq |\alpha|\,|f(x)|+|h(x)|\,|\hat{f}(x)|+|f(x)|\,|\hat{h}(x)| \leqq$$

$$\leqq |\alpha|\,|f|(x)+|h|(x)|f|\hat{\ }(x)+|f|(x)|h|\hat{\ }(x) = |\mu|\circ|f|(x).$$

Consequently, since T is an isometry,

$$\|f\|_1 = \|Tf\|_1 = \int_0^1 |\mu\circ f(x)|\,dx$$

$$\leqq \int_0^1 |\mu|\circ|f|(x)\,dx \leqq \|\mu\|\,\|f\|_1 = \|f\|_1,$$

as $M([0,1])$ is a Banach algebra with order convolution and $\|\mu\|=\|T\|=1$. Thus

$$\int_0^1 |\mu\circ f(x)|\,dx = \int_0^1 |\mu|\circ|f|(x)\,dx,$$

whence $|\mu\circ f(x)|=|\mu|\circ|f|(x)$ for almost every x in $[0,1]$. #

Lemma 2. *Let T be an isometric multiplier of $L_1([0,1])$ with order convolution, let $\mu\in M([0,1])$ be such that $Tf=\mu\circ f$, $f\in L_1([0,1])$, and let $\varphi\in AC([0,1])$ be such that $Tf=\mu\cdot f$, $f\in L_1([0,1])$, and let $\varphi\in AC([0,1])$ be such that $(Tf)\hat{\ }=\varphi\hat{f}$, $f\in L_1([0,1])$. If the multiplier S of $L_1([0,1])$ is defined by $Sf=|\mu|\circ f$, $f\in L_1([0,1])$, and $\psi\in AC([0,1])$ is such that $(Sf)\hat{\ }=\psi\hat{f}$, $f\in L_1([0,1])$, then*

$$2\mathrm{Re}\,[\overline{\varphi(x)}\,\varphi'(x)]x+|\varphi(x)|^2 = 2|\varphi'(x)|\psi(x)x+[\psi(x)]^2$$

for almost every x in $[0,1]$.

Proof. From Theorem 1 we know that $\mu=\varphi(0)\delta+\varphi'$, and so $|\mu|=|\varphi(0)|\delta+|\varphi'|$ does indeed define a multiplier S of $L_1([0,1])$ and $\psi'(x)=|\varphi'(x)|$ almost everywhere, Moreover, by Theorem 2, S is a positive multiplier and so $\psi(x)\geqq 0$ on $[0,1]$, and, by Lemma 1 and Theorem 1, for each $f\in L_1([0,1])$ we have $|(\varphi\hat{f})'(x)|=(\psi|f|\hat{\ })'(x)$ almost everywhere.

If $f(x)=1$, $0\leqq x\leqq 1$, then for almost every x in $[0,1]$ we have on the one hand

$$|(\varphi\hat{f})'(x)|^2 = |\varphi'(x)x+\varphi(x)|^2 = |\varphi'(x)|^2x^2+2\mathrm{Re}\,[\overline{\varphi(x)}\,\varphi'(x)]x+|\varphi(x)|^2,$$

whereas on the other hand

$$[(\psi|f|^\wedge)'(x)]^2 = [|\varphi'(x)|x+\psi(x)]^2 = |\varphi'(x)|^2x^2+2|\varphi'(x)|\psi(x)x+[\psi(x)]^2.$$

The conclusion of the lemma follows on equating the two identities. #

Lemma 3. *Let T be an isometric multiplier of $L_1([0, 1])$ with order convolution, let $\mu\in M([0, 1])$ be such that $Tf=\mu\circ f$, $f\in L_1([0, 1])$, and let $\varphi\in AC([0, 1])$ be such that $(Tf)^\wedge=\varphi\hat{f}$, $f\in L_1([0, 1])$. If the multiplier S of $L_1([0, 1])$ is defined by $Sf=|\mu|\circ f$, $f\in L_1([0, 1])$, and $\psi\in AC([0, 1])$ is such that $(Sf)^\wedge=\psi\hat{f}$, $f\in L_1([0, 1])$, then for almost every x in $[0, 1]$ the following statements are valid:*

(i) $\psi(x) = |\varphi(x)|$.

(ii) $|(\varphi\hat{f})'(x)| = (|\varphi|\,|f|^\wedge)'(x)$.

(iii) $|\varphi|'(x) = |\varphi'(x)|$.

(iv) $\overline{\varphi(x)}\varphi'(x) \geqq 0$.

Proof. Repeating the argument used in the proof of Lemma 2 with the functions $f(x)=(x+1)e^x$, $\hat{f}(x)=xe^x$, we see that

$$\{2\mathrm{Re}[\overline{\varphi(x)}\varphi'(x)]x+|\varphi(x)|^2(x+1)\}(x+1) = \{2|\varphi'(x)|\psi(x)x+[\psi(x)]^2(x+1)\}(x+1)$$

for almost every x in $[0, 1]$, whence, from Lemma 2, we conclude that

$$|\varphi(x)|^2x(x+1) = [\psi(x)]^2x(x+1)$$

almost every where on $[0, 1]$. Since $\psi(x)\geqq 0$ on $[0, 1]$, it follows at once that $\psi(x)=|\varphi|x)|$ almost everywhere, and part (i) is proved.

Parts (ii) and (iii) are apparent on recalling that $|(\varphi\hat{f})'(x)|=(\psi|f|^\wedge)'(x)$ and $\psi'(x)=|\varphi'(x)|$ for almost every x in $[0, 1]$.

Finally, substituting $\psi(x)=|\varphi(x)|$ in the identity of Lemma 2, we deduce

$$\mathrm{Re}[\overline{\varphi(x)}\varphi'(x)] = |\overline{\varphi(x)}\varphi'(x)|$$

almost everywhere, from which it follows that $\overline{\varphi(x)}\varphi'(x)$ is real and nonnegative for almost every x in $[0, 1]$. #

We can now prove the characterization of the isometric multipliers of $L_1([0, 1])$ alluded to at the beginning of the section.

Theorem 3. *If T is a multiplier of $L_1([0, 1])$ with order convolution, if $\mu\in M([0, 1])$ is such that $Tf=\mu\circ f$, $f\in L_1([0, 1])$, and $\varphi\in AC([0, 1])$ is such that $(Tf)^\wedge=\varphi\hat{f}$, $f\in L_1([0, 1])$, then the following are equivalent:*

(i) *The multiplier T is an isometry.*
(ii) *There exists some $\alpha\in\mathbf{C}$, $|\alpha|=1$, such that $\mu=\alpha\delta$.*
(iii) *There exists some $\alpha\in\mathbf{C}$, $|\alpha|=1$, such that $\varphi(x)=\alpha$, $0\leqq x\leqq 1$.*

Proof. Obviously part (ii) and (iii) are equivalent and imply part (i). Suppose T is an isometry. We shall show first that $\varphi'(x)=0$ almost everywhere on $[0, 1]$ and so φ is a constant since it is absolutely continuous.

If $f(x)=ie^{ix}$ and $\hat{f}(x)=e^{ix}-1$, then by Lemma 3 (iv) we have $\overline{\varphi(x)}\varphi'(x)\geqq 0$ for almost every x in $[0, 1]$, and so

$$|(\varphi\hat{f})'(x)|^2 =$$

$$= |\varphi'(x)(e^{ix}-1)+\varphi(x)ie^{ix}|^2 = 2|\varphi'(x)|^2(1-\cos x)+\overline{2\varphi(x)}\varphi'(x)\sin x+|\varphi(x)|^2 =$$

$$= 4|\varphi'(x)|^2\left(\sin\frac{x}{2}\right)^2+2\overline{\varphi(x)}\varphi'(x)\sin x+|\varphi(x)|^2.$$

Moreover, using Lemma 3 (iii) and (iv),

$$[(|\varphi||f|\hat{\ })'(x)]^2 = [|\varphi'(x)|x+|\varphi(x)|]^2 = |\varphi'(x)|^2x^2+2\overline{\varphi(x)}\varphi'(x)x+|\varphi(x)|^2.$$

Consequently, by Lemma 3 (ii)

$$4|\varphi'(x)|^2\left[\left(\sin\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)^2\right]+2\overline{\varphi(x)}\varphi'(x)[\sin x-x] = 0$$

for almost every x in $[0, 1]$.

However, an elementary calculus argument reveals that $\left(\sin\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)^2<0$, and $\sin x-x<0$, $0<x\leqq 1$, from which it follows at once that $|\varphi'(x)|^2=\overline{\varphi(x)}\varphi'(x)=0$ almost everywhere on $[0, 1]$. Thus there exists some $\alpha\in\mathbf{C}$ such that $\varphi(x)=\alpha$, $0\leqq x\leqq 1$. Furthermore, we then see that $\mu=\alpha\delta$ and so $|\alpha|=\|\mu\|=\|T\|=1$.

Therefore part (i) implies part (iii) and the theorem is proved. #

We note in passing that although very few multipliers of $L_1([0, 1])$ are isometries, it is the case that for any multiplier T of $L_1([0, 1])$ there exists a constant β such that

$$\int_0^1 Tf(y)\,dy = \beta\int_0^1 f(y)\,dy \quad (f\in L_1([0, 1])).$$

If $(Tf)\hat{\ }=\varphi\hat{f}$, $f\in L_1([0, 1])$, then $\beta=\varphi(1)$. The observation is evident on recalling that $Tf=(\varphi\hat{f})'$.

In closing, we remark that if I is any finite or infinite subinterval of the real line, then $L_1(I)$ with order convolution can be meaningfully discussed, as seen from [1, 2], and one can obtain the analogous multiplier results in the more general setting. We leave the formulation of these results to the interested reader.

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