A growth aspect of entire functions of infinite order

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1. Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be an entire function of order ϱ. We set

$$M(r) = \operatorname*{Max}_{|z|=r} |f(z)|, \mu(r,f) = \max_{n\geqq 0} \{|a_n| r^n\} \quad \text{and} \quad \nu(r,f) = \max_{n\geqq 0} \{n|\mu(r,f) = |a_n| r^n\}.$$

As usual, we call $M(r,f)$, $\mu(r,f)$ and $\nu(r,f)$ the maximum modulus, maximum term and the rank of the maximum term respectively for entire function $f(z)$. R_n is known as "*Sprungstellen*" of $\nu(r,f)$ so that $R_n < R_{n+1}$, $\nu(r)=n$ for $R_n \leqq r < R_{n+1}$. Let $M(r,f^{(s)})$, $\mu(r,f^{(s)})$ and $\nu(r,f^{(s)})$ corespond similarly to s-th derivative of $f(z)$ i.e. $f^{(s)}(z)$, where

$$f^{(s)}(z) = \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1)a_n z^{n-s}.$$

2. It is known that

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} = 0 \quad (\varrho = \infty). \quad ([4]) \tag{2.1}$$

A result better than (2.1) viz.,

$$\liminf_{r\to\infty} \frac{\log M(r)}{\nu(r)} = 0, \tag{2.2}$$

for every entire function of infinite order has been proved by SHAH [5]. SHAH and KHANNA [6] proved that for an entire function of infinite order

$$\liminf_{r\to\infty} \frac{\log \{rM(r,f^{(1)})\}}{\nu(r,f)} = 0 \tag{2.3}$$

— a result better than (2.2), since [8]

$$rM(r,f^{(1)}) > M(r)\frac{\log M(r)}{\log r}, \quad r > r_0 = r_0(f).$$

CLUNIE [2] has gone still further to prove that if s is any function of v such that $s(v)=o\left(\frac{v}{\log v}\right)$, then

$$\liminf_{r\to\infty} \frac{\log\{r^s M(r,f^{(s)})\}}{v(r,f)} = 0. \tag{2.4}$$

The object of this paper is to prove a theorem which is more precise than (2.4) and includes a result of AHMAD [1] as a special case ($s=0$).

3. In what follows, we shall prove the following theorem.

Theorem. *For an entire function of infinite order,*

$$\varliminf_{r\to\infty} \frac{\log\left[\left(r+\frac{\lambda r\log\mu(r)}{v^2(r)J(r)}\right)^s M\left(r+\frac{\lambda r\log\mu(r)}{v^2(r)J(r)}; f^{(s)}\right)\right]}{v(r)} = 0$$

where

(i) *s is a function of v such that $s(v)=o(v/\log v)$.*

(ii) *$J(r)$ is any positive function such that*

$$\sum_{m=1}^{\infty} \frac{1}{v(R_m)J(R_m)} < \infty$$

and $J(r)\to\infty$ as $r\to\infty$, λ being any fixed positive number.

PROOF. We have

$$r^s M(r,f^{(s)}) \leqq \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1)|a_n|r^n,$$

in the notations of VALIRON [7; pp. 30—31] for $n\geqq p$ $n(n-1)\dots(n-s+1)|a_n|r^n\leqq \leqq n(n-1)\dots(n-s+1)\mu(r)\ (r/R_p)^{n-p+1}$.

Therefore, we have

$$r^s M(r,f^{(s)}) \leqq \sum_{n=s}^{p-1} n(n-1)\dots(n-s+1)\mu(r)+$$

$$+\sum_{n=p}^{\infty} n(n-1)\dots(n-s+1)\mu(r)(r/R_p)^{n-p+1},$$

$$\leqq \mu(r)p^{s+1}+\mu(r)p^{2s}\left[\frac{r}{R_p-r}+\frac{r^2}{(R_p-r)^2}+\dots+\frac{r^{s+1}}{(R_p-r)^{s+1}}\right].$$

Take now

$$p = v\left(r+\frac{1}{rv^2(r)}\right)+1,$$

so that

$$R_p-1 \geqq \frac{1}{rv^2(r)}.$$

We have

$$r^s M(r, f^{(s)}) \leqq \mu(r) p^{s+1} + \mu(r) p^{2s}[r^2 \nu^2(r) + \ldots + \{r\nu(r)\}^{2s+2}]$$
$$< \mu(r) \left\{ \nu \left(r + \frac{1}{r\nu^2(r)} \right) r\nu(r) \right\}^{2s+2},$$

$$\text{(3.1)} \qquad \log \{r^s M(r, f^{(s)})\} < (1+o(1) \log \mu(r) + (2s+2) \left\{ \log \nu \left(r + \frac{1}{r\nu^2(r)} \right) \right\}.$$

Further, $\nu(r)$ is constant in the interval $R_n \leqq r < R_{n+1}$ $(n=1, 2, 3, \ldots)$, R_n tends to infinity with n or r [8, p. 30]. Then by (2.1) and our hypothesis $\varrho = \infty$, there is a subsequence of positive integers

$$\text{(3.2)} \qquad \mathbf{N}: N_1 < N_2 < \ldots < N_j < \ldots, N_j \to \infty \ (j \to \infty),$$

such that, for given an arbitrary small $\varepsilon > 0$ we have

$$\text{(3.3)} \qquad \log \mu(R_{N_j}) / \nu(R_{N_j}) < \varepsilon.$$

There are two possibilities or cases.

Case A. There is an infinite subsequence of $\mathbf{N}$, say

$$\text{(3.4)} \qquad N_{j_1} < N_{j_2} < N_{j_3} < \ldots < N_{j_k} \to \infty \quad (j_k \to \infty),$$

which we call (for convenience)

$$\text{(3.5)} \qquad \mathbf{M}: M_1 < M_2 < \ldots < M_k < \ldots \ M_k \to \infty \quad (k \to \infty),$$

and which satisfies the following condition

$$R_{M_k} + 1 > R_{M_k} + \lambda' R_{M_k} \log \mu(R_{M_k}) / \nu^2(R_{M_k}) J(R_{M_k}), \quad (\lambda' > \lambda),$$

in which case

$$\nu\big(R_{M_k} + \lambda' R_{M_k} \log \mu(R_{M_k}) / \nu^2(R_{M_k}) J(R_{M_k})\big) = \nu(R_{M_k}).$$

Case B. There is no infinite subsequence of $\mathbf{N}$ such as $\mathbf{M}$, i.e. for all large j say $j \geqq j_0$

$$R_{N_j+1} \leqq R_{N_j} \left(1 + \frac{\lambda' \log \mu(R_{N_j})}{\nu^2(R_{N_j}) J(R_{N_j})} \right),$$

in which case either $R_{N_j+1} = R_{N_j}$ and then $N_j + 1 \in N$ or $R_{N_j+1} > R_{N_j}$.

Our proof consists in showing that, in case A, Theorem is established, while, in case B, there is a contradiction which automatically rules out this case.

Case A. We take the sequence $\mathbf{M} \equiv \{M_k\}$ of (3.5) and use (3.1),

$$\log \{(R_{M_k} + \lambda R_{M_k} \beta)^s M(R_{M_k} + \lambda R_{M_k} \beta; f^{(s)})\} <$$
$$< \big(1 + o(1)\big) \left\{ \log \mu(R_{M_k}) + \int_{R_{M_k}}^{R_{M_k} + \lambda R_{M_k} \beta} \big(\nu(x)/x\big) dx \right\} + \{(2s+2) \log \nu(R_{M_k} + \lambda' R_{M_k} \beta)\},$$

where

$$\beta = \frac{\log \mu(R_{M_k})}{\nu^2(R_{M_k}) J(R_{M_k})}.$$

Since

$$s = o\left(\frac{\nu(R_{M_k})}{\log \nu\,(R_{M_k})}\right) \quad (k \to \infty)$$

therefore,

$$\frac{\log\{(R_{M_k}+\lambda R_{M_k}\beta)^s M(R_{M_k}+\lambda R_{M_k}\beta; f^{(s)})}{\nu(R_{M_k})} < (1+o(1))\left[\frac{\log \mu(R_{M_k})}{\nu(R_{M_k})} + \right.$$

$$\left. + \frac{\lambda \log \mu(R_{M_k})}{\nu^2(R_{M_k})J(R_{M_k})}\right],$$

i.e.

$$\varliminf_{k\to\infty} \frac{\log\left\{\left(R_{M_k}+\frac{\lambda R_{M_k}\log\mu(R_{M_k})}{\nu^2(R_{M_k})J(R_{M_k})}\right)^s M\left(R_{M_k}+\frac{\lambda R_{M_k}\log\mu(R_{M_k})}{\nu^2(R_{M_k}J(R_{M_k})}; f^{(s)}\right)\right\}}{\nu(R_{M_k})} = 0.$$

This leads to the desired conclusion as explained at the outset of the proof.

Case B. The proof depends on the inference that now the subsequence of integers $\mathbf{N} \equiv \{N_j\}$ defined in (3.2), beginning with (say) a certain $M_0 = N_q \geqq N_{j_0}$, consists of all integers without exception or that $\{N_j\}$ for $j \geqq q > j_0$, consists of $M_0 = N_q$, M_0+1, $M_0+2 \dots M_0+k$. It is known that

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x}\,dx.$$

Hence

$$\frac{\log \mu(R_{M_0+1})}{\nu(R_{M_0+1})} \leqq \frac{1}{M_0+1}\left\{\log \mu(R_{M_0}) + M_0 \log\left(1+\frac{\lambda' \log \mu(R_{M_0})}{\nu^2(R_{M_0})J(R_{M_0})}\right)\right\} <$$

$$< \frac{1}{M_0+1}\left\{\log \mu(R_{M_0}) + \frac{\lambda' \log \mu(R_{M_0})}{M_0 J(R_{M_0})}\right\} < \varepsilon,$$

and so $M_0+1 \in N_j$ $(j=1, 2, 3\dots)$.

Similarly,

$$M_0+2,\ M_0+3,\ M_0+4,\ \dots \in N_j.$$

Let $M_0 \in N_j$ and $M_0 > N_{j_0}$. Then

$$R_{M_0+j} \leqq R_{M_0} \prod_{n=M_0}^{M_0+j-1} \left(1+\frac{\lambda' \log \mu(R_n)}{\nu^2(R_n)J(R_n)}\right)$$

$$\leqq R_{M_0}\left(1+\frac{\varepsilon\lambda'}{\nu(R_n)J(R_n)}\right) < a \text{ constant},$$

which leads to a contradiction since R_{M_0+j} tends to infinity with j. Hence the alternative B is not possible. Thus the theorem is proved.

Remarks. 1. It is an adaptation of Shah's argument [5] combined with (3.1) that forms the basis of the proof of the theorem.

2. It is known [2] that for any entire function of finite or infinite order

$$\nu(r, f) \leqq \nu(r, f^{(1)}) \leqq \ldots \leqq \nu(r, f^{(s)}).$$

Hence it follows that $\nu(r, f)$ can be replaced by $\nu(r, f^{(s)})$ where s may have any integer value.

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References

[1] M. AHMAD, A note on entire functions of infinite order, *Compositio Math.* **17** (1968), 259—270.

[2] J. CLUNIE, Notes on integral functions of infinite order, *Quart. J. Math. Oxford Ser.* (2), **6** (1955), 88—90.

[3] Q. I. RAHMAN, A note on derivatives of integral functions, *Math. Student* **25** (1957), 21—24.

[4] S. M. SHAH, The maximum term of an entire series, *Math. Student* **10** (1942), 80—82.

[5] S. M. SHAH, The maximum term of entire series IV, *Quart. J. Math. Oxford* (2) **1** (1950), 112—116.

[6] S. M. SHAH and G. KHANNA, On entire functions of infinite order, *Math. Student* **21** (1953), 47—48.

[7] G. VALIRON, Lectures on the general theory of integral functions, *Chelsea Publishing Company* (1949).

[8] T. VIJAYARAGHAVAN, On the derivatives of integral functions, *J. London Math. Soc.* **10** (1935), 116—117.

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