The density of a sequence defined in terms of additive functions

By JÁNOS GALAMBOS *) (Philadelphia, Pa.)

Introduction

Let $f(n)$ be a real valued strongly additive arithmical function, that is, $f(nm) = f(n)+f(m)$ for coprime n and m and $f(p^a)=f(p)$ for all primes p and for all integers $a \geqq 1$. Furthermore, let $2=p_1<p_2<\dots$ be the sequence of all prime numbers and define

$$\varepsilon_k(n) = \begin{cases} 1 & \text{if } \; p_k|n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Introducing the notations

$$A_N = \sum_{p \leqq N} \frac{f(p)}{p} \qquad \text{and} \quad B_N^2 = \sum_{p \leqq N} \frac{f^2(p)}{p}, \tag{2}$$

in the present note we investigate the contribution of the largest term in the representation

$$\big(f(n)-A_N\big)/B_N = B_N^{-1} \sum_{p_k \leqq N} f(p_k)\big(\varepsilon_k(n)-1/p_k\big). \tag{3}$$

We shall obtain a result which seems to be unexpected at the first glance. Roughly speaking, the result is that, for a large class of additive functions, while each term in (3) is "negligible", the largest term is of the same magnitude as the left hand side itself.

In order to make our statements precise, let us first give some definitions.

Let $N\nu_N(n: \dots)$ denote the number of positive integers $n \leqq N$ for which the property stated in the dotted space holds.

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Definition 1. Let $h(n)$ be an arbitrary arithmetical function and let C_N and $D_N>0$ be two sequences of real numbers. We say that $(h(n)-C_N)/D_N$ has an asymptotic distribution $F(x)$ if, as $N\to+\infty$,

$$F_N(x) = \nu_N(n: h(n)-C_N<xD_N)\to F(x)$$

for all continuity points of $F(x)$.

Definition 2. A strongly additive arithmetical function $f(n)$ is said to belong to the class H (of Kubilius) if $B_N\to+\infty$ with N (see (2)) and if there is a positive integer valued function $r(N)$, tending to $+\infty$ with N, such that $\{\log r(N)\}/\log N\to 0$ as $N\to+\infty$ and $B_{r(N)}/B_N\to 1$.

For $f(n)$ from the class H, it is well known (see [1], p. 47) that, from the point of view of the existence of an asymptotic distribution of $(f(n)-A_N)/B_N$, it is sufficient to investigate the truncated sum obtained by replacing N by $r(N)$ as the upper limit for p_k on the right hand side of (3). A major step in the proof of the existence of the asymptotic distribution of this truncated sum is then to show that each term is "uniformly asymptotically negligible", that is, that for each $\eta>0$,

$$\max_{2\leqq p_k\leqq r(N)} \nu_N\big(n: |f(p_k)|\,|\varepsilon_k(n)-1/p_k| \geqq \eta B_N\big)\to 0 \tag{4}$$

as $N\to+\infty$. Our investigation is related to this fact. We shall show that for functions from the class H, while each term in (3) is uniformly small, the largest term is of the same order as $(f(n)-A_N)/B_N$ for most limiting distributions $F(x)$.

In the next section we give the exact statement of the result, together with its proof.

The result

In the theorem below we give our result where the concepts and notations of the introduction are used. In particular, $r(N)$ stands for a function occuring in Definition 2. An asymptotic distribution always means a distribution function $F(x)$, that is a non-decreasing function with $\lim F(x)=0$ as $x\to-\infty$ and $\lim F(x)=1$ as $x\to+\infty$. $F(x)$ is called degenerated at c if $F(x)=0$ for all $x\leqq c$ and $F(x)=1$ for $x>c$. The normal distribution is defined by

$$F(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Theorem. *Let $f(n)$ be a strongly additive, real valued function from the class H. Assume that $(f(n)-A_N)/B_N$ has an asymptotic distribution $F(x)$ with variance 1. Then, putting*

$$m_f(n; r(N)) = \max_{1\leqq p_k\leqq r(N)} |f(p_k)|\big(\varepsilon_k(n)-1/p_k\big),$$

$m_f(n; r(N))/B_N$ has an asymptotic distribution $M(x)$. $M(x)$ is degenerated at $x=0$ if $F(x)$ is normal and $M(x)$ is non-degenerated otherwise.

The reader is invited to compare the conclusion of the theorem with (4). As we can see, only the asymptotic normality is the exception to the rule that $m_f(n; r(N))$

is "of the same order of magnitude" as $f(n)-A_N$. For the actual asymptotic distribution $M(x)$ we shall obtain an exact formula in the course of the proof (see (5), (10) and (14)).

It is well known ([1], p. 58) that our condition concerning the existence of $F(x)$ above is equivalent to the existence of a non-decreasing function $K(u)$ of unit variation such that, at all of its continuity points, as $N\to+\infty$,

$$K_N(u) = B_N^{-2}\sum{}^* \frac{f^2(p)}{p} \to K(u) \tag{5}$$

where $\sum^*$ signifies summation over primes $p\leqq N$ for which $f(p)<uB_N$. Furthermore, $F(x)$ is normal if, and only if, $K(u)$ is degenerated at $u=0$.

PROOF of the Theorem. Let $x\geqq 0$. By definition,

$$\{m_f(n; r(N)) < xB_N\} = \{|f(p_k)|\,(\varepsilon_k(n)-1/p_k) < xB_N,\ 2\leqq p_k\leqq r(N)\} \tag{6}$$

where $\{\dots\}$ denotes sets of n satisfying the condition specified inside the brackets. If $f(p_k)=0$, then this k on the right hand side of (6) imposes no condition on n, hence we have to consider only those k for which $f(p_k)\neq 0$. Evidently, the inequality on the right hand side of (6) with $f(p_k)\neq 0$ is equivalent to

$$\varepsilon_k(n) < 1/p_k + xB_N/|f(p_k)| = a_k(f, N, x), \quad \text{say}, \tag{7}$$

and thus, by definition (1), (7) imposes no condition on n if $a_k(f, N, x)>1$ or (7) becomes $\{\varepsilon_k(n)=0\}$ if $a_k(f, N, x)\leqq 1$. Denoting by $T(f, N, x)$ the set of those primes p_k for which $f(p_k)\neq 0$ and $a_k(f, N, x)\leqq 1$, we get from (6),

$$\{m_f(n; r(N)) < xB_N\} = \{\varepsilon_k(n)=0,\ \ 2\leqq p_k\leqq r(N),\ p_k\in T(f, N, x)\}.$$

Lemma 1.4 on p. 5 of [1] thus implies

$$\nu_N(n\colon m_f(n; r(N)) < xB_N) = \big(1+o(1)\,\Pi_{T,r}(1-1/p_k)\big), \tag{8}$$

where a subscript (T, r) always signifies that the operation (here multiplication) is over elements of $F(f, N, x)$ for which $p_k\leqq r(N)$. Putting

$$L_N(x) = \sum\nolimits_{T,r} 1/p_k, \tag{9}$$

we shall show that, as $N\to+\infty$, $\lim L_N(x)=L(x)$ exists and is finite for $x>0$ and that

$$M(x) = \exp\big(-L(x)\big), \quad x>0. \tag{10}$$

We shall then complete the proof by analyzing $L(x)$.

In order to carry out this program we first observe that if $s(N, x)$ denotes the smallest element of $T(f, N, x)$, then for $x>0$, $s(N, x)\to+\infty$ with N. Indeed, by definition,

$$|f(p_k)|(1-1/p_k)\geqq xB_N \quad \text{for} \quad p_k\in T(f, N, x) \tag{11}$$

and thus by $B_N \to +\infty$, $s(N, x) \to +\infty$ as $N \to +\infty$. This observation has two immediate consequences. First of all, by

$$\log(1-z) = -z+\delta z^2, \quad |\delta| \leqq 1,\ 0 \leqq z \leqq \frac{1}{2},$$

$$\log(1-1/p_k) = -\{1+O(1/s(N, x))\}p_k^{-1} = -(1+o(1))p_k^{-1}$$

uniformly for $p_k \in T(f, N, x)$ and thus by (8) and (9)

$$\nu_N(n\colon m_f(n; r(N)) < xB_N) = (1+o(1)) \exp(-L_N(x)), \quad x > 0. \tag{12}$$

On the other hand, $s(N, x) \to +\infty$ with N implies that, instead of the criterion (11), the summation in (9) can be extended to those $p_k \leqq r(N)$, for which $|f(p_k)| \geqq x(1+o(1))$. Since for these p_k,

$$1/p_k \leqq (1+o(1))x^{-2} B_N^{-2} f^2(p_k)/p_k.$$

Definition 2 implies that for $f(n)$ from the class H, we can further extend the summation in (9) to $p_k \leqq N$ for $x>0$. We thus have for $x>0$,

$$L_N(x) = (1+o(1)) \sum\nolimits^{**} 1/p_k \tag{13}$$

where in $\sum^{**}$ summation is over primes $p_k \leqq N$, for which $|f(p_k)| \geqq x(1+o(1))B_N$. But since for any $x>0$, in view of (5),

$$\sum\nolimits^{**} 1/p_k = (1+o(1))\left\{\int_x^{+\infty} y^{-2} dK_N(y) + \int_{-\infty}^{-x} y^{-2}\, dK_N(y)\right\}$$

by the Helly—Bray theorem ([2], p. 182) and by (5) and (13), as $N = +\infty$,

$$\lim L_N(x) = L(x) = \int_x^{+\infty} y^{-2} dK(y) + \int_{-\infty}^{-x} y^{-2} dK(y), \quad x > 0 \tag{14}$$

exists and is finite. (12) and (14) thus also imply (10).

It is immediate from (8) that $M(0)=0$ and since the left hand side of (8) is non decreasing in x and is non-negative, we have that $M(x)=0$ for $x<0$ (this fact also follows from the definition of $m_f(n; r(N))$ by observing that it is non-negative for each n). From (14) we can see that $L(x)$ is non-increasing for $x>0$ and, as $x \to +\infty$, $\lim L(x)=0$. Formula (10) therefore implies that $M(x)$ is a distribution function.

Since by (14), $L(x)=0$ for all $x>0$ if, and only if, $K(y)$ is degenerated at $y=0$, the criterion quoted in connection with (5) implies that $M(x)$ is degenerated if, and only if, the limit law $F(x)$ is normal. This completes the proof.

References

[1] J. Kubilius, Probabilistic methods in the theory of numbers, Translations of Mathmeatical Monographs, *Amer. Math. Soc.*, Vol. **11** (1964).

[2] M. Loéve, Probability theory, 3rd ed. *Princeton, N. J.*, 1963.

TEMPLE UNIVERSITY
PHILADELPHIA, PA, 19122

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