On a subsequence of primes

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To the memory of Prof. A. Kertész

1. Let P denote the sequence of primes. A subsequence $\{q_n\}$ of P satisfying $3 \leqq q_1 < q_2 < \dots$ and

$$q_n \not\equiv 1 \pmod{q_i} \quad 1 \leqq i < n, n \geqq 2 \tag{1.1}$$

will be called here a G-sequence. For a sequence $B = \{b_n\}$ we denote by $A(B, x)$ the number of elements of B not exceeding x.

In [1] S. W. GOLOMB studied the density of G-sequences and he proved that there does not exist a constant $A > 0$ such that

$$A(G, x) > Ax/\log x \tag{1.2}$$

for all sufficiently large x.

A special example is the sequence G_1 defined inductively by $q_1 = 3$ and q_n for $n \geqq 2$ is the smallest prime greater than q_{n-1} for which $q_n \not\equiv 1 \bmod q_i$, $1 \leqq i \leqq n$. ERDŐS [2] proved for the sequence G_1

$$A(G_1, x) = (1 + o(1))x(\log x \cdot \log\log x)^{-1}. \tag{1.3}$$

H. G. MEIJER [3] sharped the inequality (1.2). He proved that there does not exist a constant $A > 1$ such that

$$\frac{A(G, x) \cdot \log x \cdot \log\log x}{x} > A \tag{1.4}$$

for all sufficiently large x.

Furthermore he proved that there exists a G-sequence such that

$$\overline{\lim_{x \to \infty}} \frac{A(G, x) \log x}{x} > c, \tag{1.5}$$

c being a positive constant.

Our aim is to prove the following

Theorem. *For every G-sequence we have*

$$\overline{\lim_{x}} \frac{A(G, x)}{x/\log x} < 1. \tag{1.6}$$

Furthermore, if ε is an arbitrary positive number, then there exists a G-sequence such that

$$\overline{\lim_x}\,\frac{A(G,x)}{x/\log x} > 1-\varepsilon. \tag{1.7}$$

2. Proof. The proof of (1.6) is very simple. Let $q_1<q_2<\ldots<q_N$ be the first elements of G. Then $A(G,x)$ is not greater than the number of those primes p not exceeding x for which

$$p \not\equiv 1 \pmod{q_i} \quad (i=1,\ldots,N).$$

Using the eratosthenian sieve and the prime number theorem for arithmetical progressions we get

$$A(G,x) \leqq \bigl(1+o(1)\bigr)\frac{x}{\log x}\prod_{i=1}^{N}\frac{q_i-2}{q_i-1}.$$

For $x\to\infty$, we get (1.6).

Now we prove (1.7).

Let θ be a small positive number, $\theta<\dfrac{4}{\varepsilon}$. We shall construct a sequence of positive numbers $\{x_k\}$ tending to infinity,

$$x_1<x_2<\ldots, \qquad \theta x_k > x_{k-1}$$

and a G-sequence entirely contained in the union of the intervals $(\theta x_k, x_k)$ such that (1.7) holds for this G-sequence.

For $Y>3/\theta$ let

$$L(y,\theta) = \prod_{\theta y<p<y}\left(1-\frac{1}{p-1}\right). \tag{2.1}$$

It is well known that

$$(1\geqq)\,L(y,\theta) \geqq 1-c\,\frac{\log\theta}{\log y}, \tag{2,2}$$

c being a positive constant.

Let $\varepsilon_1\geqq\varepsilon_2\geqq\ldots$ be a sequence of positive numbers such that

$$\prod_{k=1}^{\infty}(1-\varepsilon_k) > \left(1-\frac{\varepsilon}{2}\right). \tag{2.3}$$

We shall choose the sequence x_k such that

$$L(x_k,\theta) \geqq 1-\varepsilon_k \quad (k=1,2,\ldots). \tag{2.4}$$

Suppose that we have already chosen $x_1,\ldots,x_{k-1}$ $(k\geqq 1)$ and the primes $q_1<\ldots<q_N$ of the sequence G contained in

$$\bigcup_{j=1}^{k-1}(\theta x_j, x_j).$$

Let T_k be the set of primes p satisfying

$$p \not\equiv 1 \pmod{q_i} \quad (i = 1, \ldots, N). \tag{2.5}$$

The number of the primes p in the interval $\theta x < p < x$ satisfying (2.5) is asymptotically

$$(1-\theta)\frac{x}{\log x}\prod_{i=1}^{N}\frac{q_i-2}{q_i-1}. \tag{2.6}$$

Let x be so large that the number of T_k in the interval $(\theta x, x)$ is greater than

$$\left(1-\frac{3}{2}\theta\right)\frac{x}{\log x}\prod_{i=1}^{N}\frac{q_i-2}{q_i-1}.$$

Let $S_{k,x}$ be the set of those primes p' in $(\theta x, x)$ for which there exists at least one $p \in (\theta x, x)$ such that

$$p' \equiv 1 \pmod{p}. \tag{2.7}$$

The number of $S_{k,x}$ does not exceed the number of solutions of the equation

$$p'-1 = ap \quad 1 \leqq a \leqq 1/\theta, \quad p \leqq x. \tag{2.8}$$

Using the Brun—Selberg sieve method (see [4]) we see that (2.8) has at most

$$c\sum_{a<1/\theta}\frac{x}{\varphi(a)\log^2 x} = O\left(\frac{x}{\log^2 x}\log\frac{1}{\theta}\right) \tag{2.9}$$

of solutions.

If x is large enough then the right hand side of (2.9) is smaller than

$$\frac{\theta}{2}\frac{x}{\log x}\prod_{i=1}^{N}\frac{q_i-2}{q_i-1}. \tag{2.11}$$

Let now $x = x_k$ be so large that (2.5) and the other two conditions are satisfied. We continue the sequence G by all of the remained primes in $(\theta x_k, x_k)$. Hence

$$A(G, x_k) \geqq A(G, x_k) - A(G, \theta x_k) \geqq (1-2\theta)\frac{x}{\log x}\prod_{i=1}^{N}\frac{q_i-2}{q_i-1}. \tag{2.12}$$

Observing that

$$\prod_{i=1}^{N}\frac{q_i-2}{q_i-1} > \prod_{j=1}^{k-1} L(x_j, \theta) > 1-\frac{\varepsilon}{2},$$

and $(1-2\theta) \geqq 1-\frac{\varepsilon}{2}$, we get the inequality (1.7).

References

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