Problems and results on consecutive integers

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To the memory of my friend A. Kertész

In this short survey article I will discuss some questions which occupied me on and off for fourty years, some of the problems are classical other special problems. I will also prove some new results.

Denote by $f(k)$ the smallest integer so that the product of $f(k)$ consecutive integers greater than k always contain a prime greater than k. The well known theorem of Sylvester and Schur states $f(k) \leqq k$ and I proved $f(k) < \frac{3k}{\log k}$ [4]. Very much stronger results have recently been proved by JUTILA, RAMACHANDRA and SHOREY [15], they showed (improving previous results of TIJDEMAN)

$$f(k) < \frac{c_1 k \log\log\log k}{\log k \log\log k}. \tag{1}$$

(1) is certainly very far from the "truth". It seems sure that $f(k) = o(k^\varepsilon)$ and probably $f(k) < c_1(\log k)^{c_2}$ (the c's are absolute constants not necessarily the same if they have the same index). These conjectures are inaccessible at present and I have nothing to contribute towards their solution.

In [4] I prove that amongst k consecutive integers greater than k at least $k/6$ of them have a prime factor greater than k. I sharpen and extend this result, but first of all I introduce some notations. Put

$$A_k(m) = \Pi p^\alpha,\ p^\alpha \| m,\ p \leqq k$$

and denote by $f(n, k)$ the number of integers $n+i$, $1 \leqq i \leqq k$ which have at least one prime factor greater than k; $g(n, k)$ denotes the number of integers $n+i$, $1 \leqq i \leqq k$ all whose prime factors are $\leqq k$ (i.e. the number of integers $n+i$, $1 \leqq i \leqq k$ with $n+i = A_k(n+i)$). Clearly $f(n, k) + g(n, k) = k$. $U(n, k)$ denotes the number of integers $m \leqq n$ with $P(m) \leqq k$ where $P(m)$ is the greatest prime factor of m; $p(m)$ denotes the smallest prime factor of m. $g(n, k, r)$ denotes the number of integers $n+i$, $1 \leqq i \leqq r$ with $P(n+i) \leqq k$.

Theorem 1. *Let $\alpha>1$, $n>k^\alpha-k$, then*

$$f(n,k) > k\left(1-\frac{1}{\alpha}\right)-\frac{2k}{\log k}.$$

We further outline the proof of

Theorem 2. *To every $\alpha>1$ there is an $\varepsilon_\alpha>0$ so that for $k>k_0(\alpha)$ and $n>k^\alpha-k$*

$$f(n,k) > k\left(1-\frac{1}{\alpha}+\varepsilon_\alpha\right).$$

These theorems are no doubt very far from being best possible. DE BRUIJN [1] and others proved that there is a c_α so that for $k\to\infty$

$$U(k^\alpha,k) = \big(c_\alpha+o(1)\big)k^\alpha, \tag{2}$$

and in fact as $\alpha\to\infty$ c_α tends to o a little faster than $\big(([\alpha]+1)!\big)^{-1}$. It is well known and easy to see that for $1\leqq\alpha\leqq2$ $c_\alpha=1-\log\alpha$ but for $\alpha>2$ though c_α can be calculated explicitely the formula for it is quite complicated. It seems certain that for $\log n=$ $=\big(\alpha+o(1)\big)\log k$

$$f(n,k) =\big(1-c_\alpha+o(1)\big)k. \tag{3}$$

Ramachandra, Shorey and Tijdeman proved (their paper will appear in the *Journal für reine u. angew. Math.*) that if $n>\exp\big(c(\log k)^2\big)$ then $f(n,k)\geqq k-\pi(k)$ and Shorey proved (will appear in *Acta Arithmetica*) that if $n>\exp k^\varepsilon$ then $f(n,k)>$ $>k\left(1-\dfrac{ck\log\log k}{(\log k)^2}\right)$.

It seems even harder to get non trivial upper bounds for $f(n,k)$. We have

Theorem 3. *For $n\leqq k^\alpha-k$ we have*

$$f(n,k) < k(\alpha-1)+\frac{k}{\log k}.$$

For $\alpha>2$ this is trivial and I do not know any non trivial upper bound for $f(n,k)$ if $n>k^\alpha$, $\alpha>2$. I can not even prove that there is an absolute constant $c^{(\alpha)}$ so that for every $n<k^\alpha$

$$g(n,k) > c^{(\alpha)}k, \quad\text{or}\quad f(n,k) < \big(1-c^{(\alpha)}\big)k. \tag{4}$$

(4) is related to another old conjecture of mine which often annoyed me greatly: Is it true that to every α there is a $n_0(\alpha)$ so that every $n>n_0(\alpha)$ is the sum of two positive integers $a+b=n$ with $P(a\cdot b)<n^{1/\alpha}$. I have been unable to prove this for $\alpha>2$. By a heuristic independence argument one would expect that the number of solutions of

$$n = a+b,\quad P(a\cdot b) < n^{1/\alpha}$$

is $(c_\alpha^2+o(1))n$. Similarly one would expect that the events $P(n)\leqq k$ and $P(n+1)\leqq k$

are independent. More generally I conjecture that the number of integers $n<k^\alpha$ for which

$$P\left(\prod_{i=1}^{r}(n+i)\right)\leqq k \quad (r,\alpha \text{ fixed}, k\to\infty)$$

is $(c_\alpha^r+o(1))k^\alpha$. I have made no progress with this conjecture not even for $r=2$. We obtain from (2) by a simple averaging process

$$\lim_{k\to\infty}\frac{1}{rk^\alpha}\sum_{n=1}^{k^\alpha} g(n,k,r)=c_\alpha.$$

Independence arguments suggest

$$\lim_{k\to\infty}\frac{1}{r^2k^\alpha}\sum_{n=1}^{k^\alpha} g(n,k,r)^2=c_\alpha^2 \tag{5}$$

but I could not prove (5).

Denote by n_k the smallest integer with $f(n_k,k)=k$. By the *Chinese remainder theorem* we immediately obtain $n_k<\prod_{i=0}^{k-1}p_{s+i}$ where $k<p_s<p_{s+1}<\dots$ are the consecutive primes greater than k. It is easy to get a very much better upper bound for n_k. Observe that the number of integers $m<n_k$ with $P(m)\leqq k$ is at least n_k/k (since every set of k consecutive integers $\leqq n_k$ all whose prime factors are $\leqq k$). Thus we obtain from (2) by a simple computation that for $k>k_0$

$$n_k<k^{\log k/\log\log k}. \tag{6}$$

I think (6) is fairly sharp. I feel sure that for every $\varepsilon>0$ and $k>k_0(\varepsilon)$

$$n_k>\exp\left((\log k)^{2-\varepsilon}\right). \tag{7}$$

I am very far from being able to prove (7), in fact can not even show $n_k>k^{2+\varepsilon}$ which seems a ridiculously weak result. The best that I can show is

$$n_k>k^2\exp\left((\log k)^c\right)$$

for a certain $c>0$.

I once conjectured that for $n\geqq 2k$ $\binom{n}{k}$ is always divisible by at least one of the integers $n-i$, $0\leqq i<k$. SCHINZEL disproved this conjecture and our results with SCHINZEL [18] makes it likely that there is a k_0 so that for $k>k_0$ the conjecture only holds if $k=p^\alpha$. I now conjecture that there is a $c<1$ so that for $n\geqq 2k$ $\binom{n}{k}$ always has a divisor d satisfying $cn<d\leqq n$.

Several mathematicians investigated $P\left(\binom{n}{k}\right)$ ([5]). I conjecture that for every $\alpha<\frac{1}{2}$ and $k>k_0(\alpha)$

$$P\left(\binom{n}{k}\right)>\min\{\exp(c_\alpha k^\alpha),\ \exp c_\alpha(\log n)^\alpha\}. \tag{8}$$

The results of RANKIN and myself [6] imply that for $k<c\log n$

$$\liminf P\left(\binom{n}{k}\right)\Big/\log n = 0,$$

but perhaps if $\alpha>0$ then

$$P\left(\binom{n}{k}\right) > n^{c_\alpha}$$

for every $k>(\log n)^\alpha$ and $c_\alpha \to 1$ as $\alpha\to 1$.

It is known that $P(n(n+1))>c\log\log n$ and perhaps

$$\lim_{n\to\infty} P(n(n+1))/(\log n)^{2-\varepsilon} = \infty \tag{9}$$

for every $\varepsilon>0$. (9) if true is certainly very deep. On probability grounds I expect (by (2)) that for every $\varepsilon>0$ and infinitely many n

$$P(n(n+1)) < (\log n)^{2+\varepsilon}. \tag{10}$$

(10) if true is no doubt very deep. I can not even prove that

$$\liminf \log P(n(n+1))/\log n = 0.$$

Let $h(k)$ be the greatest integer so that for every $n\geqq h(k)$

$$P\left(\prod_{i=1}^{k}(n+i)\right) > h(k).$$

It is not difficult to prove $h(k)>ck\log k$ but no doubt $h(k)$ increases much faster and it seems very difficult to get a good estimation for $h(k)$. $h(k)>k^c$ is certainly true for every c if $k>k_0(c)$.

$$h(k)/k\log k \to \infty$$

follows from (1).

A well known conjecture of Cramer states

$$\limsup \frac{p_{n+1}-p_n}{(\log n)^2} = 1 \tag{11}$$

where $p_1<p_2<\dots$ is the sequence of consecutive primes. If (11) is true the order of magnitude of $h(k)$ probably will be $\exp((\log k)^2)$. Let in fact p_{n_k} be the smallest prime for which $p_{n_k+1}-p_{n_k}>k$. Clearly

$$h(k)\leqq P\left(\prod_{i=1}^{k}(p_{n_k}+i)\right).$$

The problems on $h(k)$ are of course connected with the conjecture (8). Related problems were investigated by ECKLUND, EGGLETON and SELFRIDGE [2].

For fixed n

$$P\left(\prod_{i=1}^{k}(n+i)\right) = u_n(k)$$

is clearly a non decreasing function of k. Denote by $p^{(n)}$ the least prime greater than n. Clearly $u_n(k)$ ceases to be interesting for $k \geqq p^{(n)}-n$, since it then equals the greatest prime $\leqq n+k$. It might be of some interest to determine the maximum number of possible increases of $u_n(k)$, $1 \leqq k < p^{(n)}-n$ as a function of n — can it be as big as $\log\log n$? or $\log n$? The answer to the first question is probably yes to the second one no.

$$P\Big(\prod_{j<p_{i+1}-p_i}(p_i+j)\Big) = P_i$$

also is an interesting function. I do not think that P_i tends to infinity, in fact perhaps $P_i=3$ for infinitely many i in other words there are infinitely many integers $n=2^\alpha 3^\beta$ for which $n+1$ and $n-1$ are both primes. On the other hand for most values of i P_i tends to infinity probably quite fast — the explanation of my vagueness is ignorance.

By a simple averaging process I deduced [8]

$$\min_{1 \leqq i \leqq k} A_k(n+i) < ck. \tag{12}$$

I conjectured that (12) holds for every $c>0$ if $k>k_0(c)$. This conjecture always seemed very interesting to me, but unfortunately I was unable to make any progress with it. More generally it would be interesting to investigate

$$A(k) = \max_n \min_{1 \leqq i \leqq k} A_k(n+i).$$

$A(k) \to \infty$ as $k \to \infty$ is not hard to prove but I have no idea how fast $A(k)$ tends to infinity. A stronger conjecture than (12) is

$$\sum_{i=1}^{k} \frac{1}{A_k(n+i)} \to \infty \quad \text{as} \quad k \to \infty$$

perhaps even

$$\sum_{i=1}^{k} \frac{1}{A_k(n+i)} \geqq \big(1+o(1)\big)\log k.$$

It is easy to deduce by Turán's method [19] (using second moments) that the normal order of

$$\sum_{i=1}^{k} \frac{1}{A_k(n+i)} \quad \text{(as } n \text{ varies) is} \quad \big(1+o(1)\big)e^{-c}\,\frac{\pi^2}{6}\,\frac{k}{\log k}$$

where c is Euler's constant. Clearly

$$\min_n \max_{1 \leqq i \leqq k} A_k(n+i) = k.$$

I do not know how large is the normal order of this function. Probably it is more than k^c for every c if $k \gg k_0(c)$. A well known result of MAHLER [17] implies that for every k and $\varepsilon>0$

$$A_k\left(\prod_{i=1}^{k}(n+i)\right) < n^{1+\varepsilon}. \tag{13}$$

On the other hand it is easy to see that for infinitely many n

(14) $$A_3(n(n+1)) > cn\log n.$$

It would be very interesting to sharpen (13) and (14) but I could not even prove that for some fixed k

$$\limsup_{n\to\infty} A_k\left(\prod_{i=1}^{k}(n+i)\right)\Big/ n\log n = \infty.$$

By an averaging process it is not difficult to prove that

$$\lim_{k\to\infty} A_k\left(\binom{n}{k}\right)^{1/k} = c$$

holds for almost all n and large k.

Pleasant proved that $v\left(\binom{n}{k}\right) \geqq v(n)$ always holds (unpublished) $(v(n)$ denotes the number of distinct prime factors of $n)$. Selfridge and I conjectured [7]

$$\max_{1\leqq k\leqq n} v\left(\binom{n}{k}\right)\Big/ k \to \infty \quad \text{as} \quad n\to\infty$$

but we have not even proved

(15) $$\max_{1\leqq k\leqq n}\left(v\left(\binom{n}{k}\right)-k\right) \to \infty \quad \text{as} \quad n\to\infty.$$

(15) follows from recent results of Shorey and Ramachandra (will appear in *Acta Arithmetica*).

Let $\alpha<1$, $k=n^{\alpha+o(1)}$. Very likely

(16) $$v\left(\binom{n}{k}\right) = (1+o(1))k\sum_{n^\alpha}^{n}\frac{1}{p} = (1+o(1))k\log\frac{1}{\alpha}.$$

(16) if true is certainly deep.

Put $t_1=(\log n)^{1-\varepsilon}$, $t_2=(\log n)^{1+\varepsilon}$. It is easy to see from the Chinese remainder theorem that

$$\limsup_{n\to\infty}\frac{1}{t_1\log\log n}\sum_{i=1}^{t_1} v(n+i) = \infty.$$

Very likely

(17) $$\lim_{n\to\infty}\frac{1}{t_2\log\log n}\sum_{i=1}^{t_2} v(n+i) = 1.$$

I have no idea how to attack (17), perhaps (17) already holds for $t_2=\log n$, for some related questions see [7].

I proved [9] that

$$\max_{1\leqq k\leqq \frac{n}{2}} v\left(\binom{n}{k}\right) = (1+o(1))\frac{n\log 2}{\log n}$$

and the maximum is assumed for $k=(1+o(1))\frac{n}{2}$. As k increases from 1 to $\frac{n}{2}$, $v\left(\binom{n}{k}\right)$ has of course a tendency to increase, but no doubt for every c and $n>n_0(c)$ there are values of k $\left(1\leqq k<\frac{n}{2}\right)$ for which

$$v\left(\binom{n}{k}\right) > v\left(\binom{n}{k+1}\right)-c.$$

GRIMM [14] conjectured that if $n+1, \dots, n+t$ are consecutive composite numbers then for $1\leqq i\leqq t$ there is a $p_i|n+i$, $p_i\neq p_j$ for $i\neq j$. SELFRIDGE and I [10] proved this for $t=o(\log n)$, we also showed that Grimm's conjecture if true is certainly very deep. Ramachandra, Shorey and Tijdeman proved that Grimm's conjecture holds for $t=[c(\log n/\log\log n)^3]$. Their paper will soon appear in *J. reine u. angew. Math.* For further problems and results in this direction I have to refer to the original papers which are quoted in the paper of Ramachandra, Tijdeman and Shorey.

Selfridge and I proved that the product of consecutive integers is never a power. Our proof will soon appear in the *Illinois Journal of Mathematics*. The following problem is of interest here: Denote by $H_k(n, l)$ the smallest integer i_k for which

$$(n+i_1)\dots(n+i_k) = x^l,\ 0 \leqq i_1 <\dots< i_k = H_k(n, l)$$

is solvable. Our theorem with Selfridge asserts $H_k(n, l)\geqq k$ for every $k>1$, $n\geqq 1$ and $l>1$.

There seems little doubt that this can be improved a great deal, but as far as I know this question has not yet been seriously investigated.

Selfridge and I in fact prove that for every k and $l>1$ $\prod_{i=1}^{k}(n+i)$ has a prime factor $p\geqq k$ with $p^\alpha \| \prod_{i=1}^{k}(n+i)$, $\alpha\not\equiv 0 \pmod{l}$. We conjecture that in fact for $k>2$ there is such a prime with $\alpha=1$, we made no progress with this deep conjecture. I once thought that for every $k>1$ $\prod_{i=1}^{k}(n+i)$ has a prime p for which $p\| \prod_{i=1}^{k}(n+i)$, but Mahler pointed it out to me that since $x^2-8y^2=1$ has infinitely many solutions there are infinitely many values of n for which every prime factor of $n(n+1)$ occurs with an exponent greater than one. I would guess though that the number of such integers $n<x$ is less than $(\log x)^c$. Probably there always is a p with $p^\alpha\| n(n+1)$, $\alpha\leqq 2$, in fact if $a_1<a_2<\dots$ is the sequence of integers all whose prime factors are $\geqq 3$ then one would conjecture $\lim_{i\to\infty}(a_{i+1}-a_i)=\infty$, and in fact $a_{i+1}-a_i>c_1 i^{c_2}$.

Put

$$S_r(m) = \prod p^\alpha,\ p^\alpha\| m,\ \alpha > r.$$

Let k be fixed. Determine or estimate

$$\limsup_{n\to\infty} \log S_1\left(\prod_{i=1}^{k}(n+i)\right)\Big/\log n = \alpha_k.$$

Trivially $\alpha_k\leqq k$ and by Mahler's remark $\alpha_k\geqq 2$. It would be interesting to improve these bounds. It is fairly sure that for $k>2$, $\alpha_k<k$ and I would not be surprised if $\alpha_k=2$ for every k.

Observe that

$$\frac{m}{S_1(m)} = \prod_{p \| m} p.$$

RIGGE [13] and a few month later I proved in 1939 that the product of consecutive integers is never a square. Our proof was based on the fact that for $k>4$ the k integers $A_k(n+i)/S_1(A_k(n+i))$, $i=1, \ldots, k$, can not all be different. Two strengthenings of this result: Is it true that for $r>1$ there is a $k_0(r)$ so that for $k>k_0(r)$ the k integers $A_k(n+i)/S_r(A_k(n+i))$ can not all be different? I have not decided this even for $r=2$. Is there a $c<1$ so that for $k>k_0(c)$ the ck integers $A_k(n+i)/S_1(A_k(n+i))$, $1 \leqq i \leqq ck$, can not all be different?

Let $h(n)$ be the largest integer for which the integers $\dfrac{n+i}{S_1(n+i)}$, $i=1, \ldots, h(n)$ are all distinct. It is very likely that infinitely often $h(n)>cn^{\frac{1}{3}-\varepsilon}$ but I do not see at present whether $h(n)/n^{1/2} \to 0$ is true or false.

Denote by $G(k)$ the largest integer for which there are $G(k)$ consecutive integers $n+i$, $1 \leqq i \leqq G(k)$ for which the integers $A_k(n+i)$, $1 \leqq i \leqq G(k)$ are all different. BASIL GORDON and I proved some time ago the following

Theorem 4. *Let $2=p_1<p_2<\ldots<p_s \leqq k<p_{s+1}<p_{s+2}$ be the sequence of consecutive primes. Then*

$$p_{s+2}-2 \leqq G(k) \leqq (2+o(1))k.$$

We have no counterexample to $G(k)=p_{s+2}-2$. Since our proof has never been published I give it here in full detail.

Before we prove our Theorems we state a few disconnected problems on consecutive integers. Selfridge and I conjectured that for $n \geqq 2k$ $\binom{n}{k}$ has a prime factor $<\frac{n}{2}$ the sole exception is $\binom{7}{3}$. Ecklund proved this conjecture [3]. Selfridge and I proved that there is an absolute constant c so that for $n \geqq 2k$ and $k>k_0(c)$ $\binom{n}{k}$ has a prime factor $<\frac{n}{k^c}$ [10]. Selfridge and I conjectured that perhaps c can be taken as 1. Very likely this is best possible since we almost certainly have

$$\limsup_{n \to \infty} p\left(\binom{n}{k}\right)\Big/ n = \frac{1}{k}. \tag{18}$$

(18) if true will certainly be very hard to prove. Ecklund, Selfridge and I investigated the smallest integer $m_k \geqq k+1$ for which all prime factors of $\binom{m_k}{k}$ are greater than k [12], our lower bounds for m_k are poor, we only get $m_k>k^{1+c}$.

In a forthcoming paper in *Mathematics of Computation* GRAHAM, RUZSA, STRAUS and I prove that to every two odd primes p and q there are infinitely many values of n for which $p \nmid \binom{2n}{n}$ and $q \nmid \binom{2n}{n}$. We could not extend this for three primes.

We also state the following *problem:* Is it true that there is an absolute constant c so that

$$\sum \frac{1}{p} < c \quad \text{where} \quad p \nmid \binom{2n}{n} \quad \text{and} \quad p < 2n?$$

Many rather special problems can be stated on binomial coefficients — here are a few which occurred me recently. Denote by α_r, $r=0, 1, \ldots$ the density of integers n for which there are exactly r squarefree integers in the sequence $\binom{n}{k}$, $1 \leqq k \leqq n-1$. It is not difficult to prove that α_r exists and $\sum_{r=0}^{\infty} \alpha_r = 1$. The following question seems much more difficult: Denote by $s(n)$ the number of squarefree integers amongs the sequence $\binom{n}{k}$, $1 \leqq k \leqq n-1$. Probably $s(n)$ can not be very large, perhaps $s(n) = o(n^{\varepsilon})$ for every $\varepsilon > 0$, but I have not obtained any sharp results.

The prime number theorem implies

$$\max_{1 \leqq m \leqq n} v(m) = (1+o(1)) \frac{\log n}{\log \log n}.$$

It is very likely that for every k

$$\max_{1 \leqq m \leqq n} v\left(\prod_{i=1}^{k} (m+i)\right) = (1+o(1)) \log n / \log \log n. \tag{19}$$

(19) if true will be very hard to prove. Put ($d(n)$ is the number of divisors of n)

$$D(n) = \max_{1 \leqq m \leqq n} d(m).$$

$D(n)$ has been studied among others by RAMANUJAN in his paper "On highly composite numbers" (see Collected papers of S. RAMANUJAN, Cambridge, 1927). Almost certainly

$$\limsup_{n \to \infty} \max_{1 \leqq m \leqq n} d\big(m(m+1)\big)/D(n) = \infty$$

but probably for a certain $c > 0$

$$\max_{1 \leqq m \leqq n} d\big(m(m+1)\big) < n^c D(n).$$

All these questions seem very difficult. On the other hand it is a simple exercise to prove that for every k ($\sigma(n)$ is the sum of the divisors of n)

$$\max_{1 \leqq m \leqq n} \frac{\sigma\left(\prod_{i=1}^{k} (m+i)\right)}{\prod_{i=1}^{k} (m+i)} - \max_{1 \leqq m \leqq n} \frac{\sigma(m)}{m} \to 0.$$

Put $A_k = \prod_{i=1}^{k} p_i$, where p_i are the consecutive primes. The prime number theorem implies $A_k^{1/p_k} \to e$. Denote by $g(k)$ the number of integers $m < A_k$ for which

$(m+i, A_k)>1$ for $i=1, 2, \ldots, p_k$. $g(k)$ certainly tends to infinity with k but perhaps

$$\lim_{k\to\infty} g(k)^{1/p_k} < e?$$

Many similar questions can be asked which are connected with the growth of $p_{i+1}-p_i$. Selfridge and I asked: Is it true that to every r there is a $k_0(r)$ so that for $k>k_0(r)$ there is an $m<A_k$ with $v\big((m+i, A_k)\big)>r$ for $i=1, 2, \ldots, p_k$? This problem seems to be surprisingly difficult and is perhaps not affirmative for every r. A related question states as follows:

Denote by $K(n)$ the largest integer so that for every $1\leqq i\leqq K(n)$ $v(n+i)>\log\log n$. The Chinese remainder theorem implies $K(n)\geqq\big(1+o(1)\big)\log n/(\log\log n)^2$ and this is the best lower bound that I can get. I have no non trivial upper bound for $K(n)$. On probability grounds one would expect

$$\limsup_{n\to\infty} K(n)/\log n = \frac{1}{\log 2}.$$

An old conjecture of Catalan states that 8 and 9 are the only consecutive powers. This problem seemed intractable. A few months ago TIJDEMAN proved that there is a computable constant c so that two consecutive integers greater than c can not be both powers and CHOODNOVSKI proved that there is an explicitely given function $L(n)$ so that if $1=a_1<a_2<\ldots$ is the sequence of powers then $a_{n+1}-a_n>L(n)$ holds for every n.

Now we prove our Theorems. Clearly $\binom{n}{k}>\frac{n^k}{k^k}$, also a well known lemma asserts that $p^\beta \| \binom{n}{k}$ implies $p^\beta\leqq n$. Now put $\binom{n}{k}=u_1\cdot u_2$ where $P(u_1)\leqq k$ and $p(u_2)>k$. From $\pi(k)<\frac{2k}{\log k}$ we obtain $u_1<n^{2k/\log k}$. Thus

$$n^{f(n-k,k)} > u_2 > n^{k-\frac{2k}{\log k}}\cdot k^{-k}. \tag{20}$$

(20) immediately implies Theorem 1 by $n>k^\alpha$.

It might be of some interest to investigate

$$\min_{p^\alpha \| \binom{n}{k}} p^\alpha = V(n,k) \quad \text{and} \quad \min_{\substack{p^\alpha \| \binom{n}{k} \\ p\leqq k}} p^\alpha$$

and try to obtain upper and lower bounds for these functions. I had no time to investigate whether one can get any non trivial results.

We now outline the proof of Theorem 2. Let $n-k+1\leqq m_1<\ldots<m_{f(n-k,k)}\leqq n$ be the integers which have a prime factor greater than k. In the proof of Theorem 1 we crudely assumed that the m_i are entirely composed of the primes greater than k and obtained Theorem 1 from (20). In fact we can improve this estimate by the following

Lemma 1. *To every $\eta_1>0$ there is an $\eta_2>0$ so that for $k>k_0(\eta_1,\eta_2)$ all but $\eta_1 k$ of the integers $n-k+1\leqq m\leqq n$ have a prime factor p satisfying $k^{\eta_2}<p\leqq k$.*

The proof of Lemma 1 follows easily by the *sieve of Eratosthenes* and will be left to the reader.

From Lemma 1 we obtain that at least $f(n-k, k)-\eta_1 k$ of the integers m_i have a prime factor p satisfying $k^{\eta_2}<p\leqq k$. Thus (20) can clearly be replaced by the following sharper inequality

$$n^{f(n-k,k)} > n^{k-\frac{2k}{\log k}}\cdot k^{-k}\cdot k^{\eta_2(f(n-k,k)-\eta_1 k)}. \tag{21}$$

Using $n\geqq k^{\alpha}$ (21) easily implies Theorem 2.

Theorem 3 is nearly trivial. We evidently have $\left(\text{since } \binom{n}{k} \text{ is an integer}\right)$

$$\prod_{i=0}^{k-1}{}' (n-i) \geqq k! \tag{22}$$

where the dash indicates that we remove from $n-i$ all prime factors greater than k. From (22) and $n\leqq k^{\alpha}$ we have

$$k^{k\alpha} \geqq n^k > k!\, k^{f(n-k,k)} > k^{k+f(n-k,k)}e^{-k}$$

which proves Theorem 3.

The most difficult proof is our proof with GORDON of Theorem 4. First we observe $G(k)\geqq p_{s+2}-2$. Clearly the $p_{s+2}-2$ integers $A_k(2), A_k(3), \ldots, A_k(p_{s+2}-1)$ are all distinct since $A_k(i)=i$ for all of them except for $A_k(p_{s+1})$ which is 1. The proof of the upper bound of Theorem 4 is a little more difficult. Suppose that the $G(k)$ integers $A_k(n+i)$, $1\leqq i\leqq G(k)$ are all different. Observe that

$$\Pi' A_k(n+i) \geqq G(k)!\,G(k)^{-\pi(k)} > G(k)^{G(k)}\cdot e^{-G(k)}\cdot G(k)^{-\pi(k)}, \tag{24}$$

where in $\Pi' A_k(n+i)$, $1\leqq i\leqq k$, we omit for each $p\leqq k$ the integer $n+i$ which is divisible by the highest power p^{α} (if there are several such $(n+i)$'s we omit the greatest one). (24) follows immediately since by assumption the $A_k(n+i)$, $1\leqq i\leqq G(k)$ are all distinct. On the other hand

$$\Pi' A_k(n+i) \leqq G(k)!\,\Big(\prod_{k<p\leqq G(k)} p\Big)^{-1} < G(k)^{G(k)}\cdot e^{-2G(k)+k+o(G(k))}. \tag{25}$$

The first inequality of (25) follows from the fact that by Legendre's formula all primes $p\leqq k$ occur in a higher power in $G(k)!$ then in $\Pi' A_k(n+i)$ and by definition $A_k(n+i)$ has no prime factor greater than k; the second inequality of (25) follows from Stirling's formula. (24) and (25) implies

$$e^{-G(k)+k+o(G(k))} > G(k)^{-\pi(k)}$$

which by the prime number theorem implies $G(k)\leqq(2+o(1))k$ and hence the proof of Theorem 4 is complete. It seems very likely that the upper bound in Theorem 4 can be improved but we have not been successful in our attempts.

To finish this paper I state one more *problem:* Determine or estimate the smallest integer $H(n)$ so that one can find two subsets of the integers $n+1, n+2, \ldots, n+H(n)$ whose product is equal.

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