Ideals primary relative to a radical

By W. G. LEAVITT (Lincoln, Nebraska)

Dedicated to the memory of Andor Kertész

1. Introduction

Let P be an arbitrary (Kurosh—Amitsur) radical class of rings with $P(R)$ designating the P-radical of a ring R. For basic definitions and notation we refer to [1]. We will let N denote the class of all nil rings; J the class of all Jacobsen radical rings; and B the lower Baer radical class, that is B is the smallest radical containing all nilpotent rings. Note that if for a class M of rings we let $UM=\{R|every\ 0\neq R/I\notin M\}$ then, alternatively, $B=UA$ where A is the class of all prime rings (that is, B is the *upper radical* defined by the class of all prime rings).

A ring R will be called *left P-primary* if $IJ=0$ for ideals I, J of R implies either $J=0$ or $I\subseteq P(R)$. A *right P-primary* ring is defined symmetrically, and R is said to be *2-sided P-primary* if it is both left and right P-primary. That is, if we let Q_l, Q_r, and Q denote respectively the classes of all left, right, and 2-sided P-primary rings then $Q=Q_l\cap Q_r$. Throughout the paper there will be definitions, statements and theorems which will be equally valid whichever of these three "primaries" is chosen and when this is the case we will write simply *P-primary*. For example, we will say that an ideal I of a ring R (which we will write $I\lhd R$) is *P-primary* if R/I is a P-primary ring, which means that I is a left P-primary ideal if R/I is a left P-primary ring, and similarly for I right or 2-sided P-primary. For an arbitrary $I\lhd R$ we will also define $G(I)$ to be the ideal of R defined by $G(I)/I=P(R/I)$. Note here, for later use, that if $A\subseteq B$ where A and B are ideals of R then $G(A)\subseteq G(B)$.

When R is commutative it is, of course, true that the three "primaries" coincide and if $P=N$ then a P-primary ideal I of a commutative ring R is "primary" in the classical sense of E. NOETHER [see 2]. That is, if $xy\in I$ and $y\notin I$ then $x^n\in I$ for some integer $n\geqq 1$. Thus our theory is a generalization of the classical theory and we will show that certain of the classical theorems can be generalized, particularly for radicals which contain B or are hereditary (a radical P is hereditary if $I\lhd R\in P$ implies $I\in P$). Our proofs will in some cases be similar to the proofs of the classical theory. However, there are certain differences and in any case we will, for completeness, at least sketch the proof. One other point to be remarked is that in most cases the proofs for right or 2-sided P-primary exactly parallels that for left P-primary and when this is so we will present only the left P-primary proof.

As for prime ideals we have alternative characterizations of P-primary ideals by:

Proposition 1. An ideal I of R is left P-primary if and only if for left (right) ideals A, B or R, whenever $AB \subseteq I$ with $B \not\subseteq I$ then $A \subseteq G(I)$.

PROOF. Let A, B be left ideals of R with $AB \subseteq I$. Then $(A+AR)(B+BR) \subseteq I$. Thus if I is left P-primary and $B \not\subseteq I$ then $A+AR \subseteq G(I)$ so $A \subseteq G(I)$. The converse is obvious, as is the proof for right ideals (as well as the analogous proofs for the right or 2-sided P-primary cases).

Proposition 2. An ideal I of R is left P-primary if and only if $xRy \subseteq I$ where $x, y \in R$, implies either $y \in I$ or $x \in G(I)$.

PROOF. Clear.

Remark 1. If P is the zero radical, that is $P=\{0\}$, then an ideal I of a ring R is a P-primary ideal if and only if it is a prime ideal. On the other hand, all prime ideals are P-primary for all radicals P. Also notice that for R an arbitrary ring $G(P(R))=P(R)$ hence also $G(G(I))=G(I)$ for any $I \lhd R)$ and so $P(R)$ is a P-primary ideal if and only if it is a prime ideal.

In all of the cases noted in Remark 1 all three P-primaries coincide. However, this is not true in general, and in the following example we will construct for $P=N$ a ring which is left but not right P-primary.

Example 1. Let R be the ring (with unit) of all polynomials over a field F in non-commuting variables x, y with relations $x^2=xy=0$. We can write $R=\{f_1+f_2x \mid f_1, f_2 \in F[y]\}$. Now $xRy=0$ with $x \neq 0$ and y non-nilpotent so $y \notin P(R)$. Thus R is not right P-primary. On the other hand, let $fRg=0$ for some $f=f_1+f_2x$ and $g=g_1+g_2x$. Then $fg=0$ and if $g \neq 0$ either $g_1 \neq 0$ which implies $f_1=0$ or else $g_1=0$ but $g_2 \neq 0$, and this also implies $f_1=0$. But then $f=f_2x$ and clearly $(x) \subseteq P(R)$. Thus R is left P-primary by Proposition 2.

Remark 2. In this example it is clear, in fact, that if $fRg=0$ with $g \neq 0$ then f is a member of a nilpotent ideal so that $f \in B(R)$. Therefore R is left P-primary for all radicals $P \supseteq B$. On the other hand, for any radical P such that $y \notin P(R)$ the ring is not right P-primary. This includes any radical P such that $B \subseteq P \subseteq J$ and from our example we conclude that $Q_r \neq Q_l$ for any such radical. This does, however, still leave open the question of whether or not this is so for an arbitrary (non-trivial) radical.

Remark 3. We should record one further situation in which all three P-primaries coincide: If $R \in P$ then R and all its ideals are automatically P-primary.

2. Intersections of P-primary ideals

Since the classical primary refers to N which is a hereditary radical, it would be expected that more of the classical theory would apply to a hereditary radical than to radicals in general. Thus we have:

Theorem 1. *If P is a hereditary radical and A, B ideals of a ring R then $G(A \cap B) = G(A) \cap G(B)$.*

PROOF. Since $A \cap B$ is contained in both A and B we have in any case (whether P is hereditary or not) that $G(A \cap B) \subseteq G(A) \cap G(B)$. Write $G(A)=G$ and $G(B)=H$. Now $(G \cap H+A)/A \lhd G/A$ so by the hereditary property $(G \cap H+A)/A \cong (G \cap H)/ /(A \cap H) \in P$. Simitarly, $(A \cap H+B)/B \lhd H/B$ so $(A \cap H+B)/B \cong (A \cap H)/(A \cap B) \in P$. Since radicals are closed under homomorphic extensions, this implies $G \cap H/A \cap B \in P$ and so $G \cap H \subseteq G(A \cap B)$.

As in the classical case this leads to a useful property of P-primary ideals:

Theorem 2. *If P is a hereditary radical and A, B are P-primary ideals of a ring R such that $G(A)=G(B)$ then $A \cap B$ is also a P-primary ideal with $G(A \cap B)= =G(A)=G(B)$.*

PROOF. We only need show P-primariness so suppose A, B left P-primary with $CD \subseteq A \cap B$ for some ideals C, D of R. If $D \nsubseteq A \cap B$ then, say, $D \nsubseteq A$ which implies $C \subseteq G(A)=G(A \cap B)$.

Remark 4. This proof actually shows that if A, B are P-primary and $G(A)= =G(B)=G(A \cap B)$ then $A \cap B$ is also P-primary whether P is hereditary or not. However, when P is non-hereditary, even if $G(A)=G(B)$, it is generally true that $G(A \cap B)$ is properly contained in $G(A)$, and we will give an example of such a case below.

First, though, we will need to recall the Kurosh construction of the lower radical LM for an arbitrary class M of rings. Let M_1 be the homomorphic closure of M and for β an arbitrary ordinal $M_\beta = \{|$every $0 \neq R/I$ has a non-zero ideal in M_α for some $\alpha < \beta\}$, then $LM = \bigcup_\beta M_\beta$ is the Kurosh lower radical of M (that is, the smallest radical class containing M). It was shown [2, p. 618] that if all rings in M_1 are idempotent then $LM=M_2$. As a corollary we have: *If M is a class of rings with unit then $LM=M_2$.*

Proposition 3. Let $P=LM$ where M is a class of rings with unit. (1) *If a ring R has radical $P(R) \neq 0$ then $R=U \oplus V$ where U and V are ideals of R with $U \in M_1$.* (2) *If R is a ring with unit containing no non-zero orthogonal idempotents then $P(R) \neq 0$ implies $R \in M_1$.*

PROOF. (2) is obvious from (1) so suppose R is a ring with $P(R) \neq 0$ then R has a non-zero ideal $I \in M_2$. But then we have $0 \neq U \lhd I$ with $U \in M_1$. Since U is idempotent it follows that $U \lhd R$ and since it has a unit it is a direct summand of R.

Example 2. We will construct a (necessarily non-hereditary) radical P and a ring K containing ideals A, B such that $G(A)=G(B)=K$ (so A and B are P-primary) but $G(A \cap B)$ is a proper ideal of K.

Let $Z_2[x, y, z]$ be the ring of polynomials (commuting variables) over the field Z_2 of integers mod 2, and let $K=Z_2[x, y, z]/(z^2)$. Define $R=K/(x) \cong K/(y)$ and let $P=L\{R\}$ which satisfies the hypothesis of Proposition 3. Since both $K/(x)$ and $K/(y)$ are in P it follows that both (x) and (y) are P-primary with $G(x)=G(y)=K$. Now $(x) \cap (y)=(xy)$ and we let $H=K/(xy) \cong Z_2[x, y, z]/(xy, z^2)$. We can write $H=\{f_1+f_2z+yg_1+yg_2z\}$ where $f_1, f_2 \in Z_2[x]$ and $g_1, g_2 \in Z_2[y]$, and from this it easily follows that the only idempotent in H is its unit. From Proposition 3 it follows that if $P(H) \neq 0$ then $H \in M_1$.

Now to see what members of M_1 look like, let I be a proper ideal of $R \cong \cong Z_2[y, z]/(z^2) = \{f_1 + f_2 z\}$ where $f_1, f_2 \in Z_2[y]$. Let $f \in Z_2[y]$ be the polynomial of smallest degree such that $fz + h \in I$ for some $h \in Z_2[y]$ and let $g \in Z_2[y]$ be the polynomial of smallest degree such that $g \in I$. It is easy to see that $I = (fz + h, g)$ so that R/I is a finite ring. But H is an infinite ring so $H \not\cong R/I$ for any proper ideal I of R. Also $H \not\cong R$ since H has non-nilpotent zero divisors whereas R does not. Thus $H \notin M_1$ so we conclude that $P(H) = 0$, that is $G(xy) = (xy) \neq K$.

Remark 5. Note that in this example $(z)^2 = 0$ with $z \notin (xy)$ so $(x) \cap (y)$ is not a P-primary ideal of K. It would seem probable that (for some P) a ring could exist containing ideals A, B such that A, B and $A \cap B$ are all P-primary but $G(A \cap B) \neq \neq G(A) \cap G(B)$.

3. The uniqueness theorems

An ideal I of a ring R is said to have a *P-primary representation* if $I = I_1 \cap \ldots \cap I_n$ for some finite set $\{I_i\}$ of P-primary ideals of R. The representation is said to be *proper* if the intersection is irredundant and all $G(I_i)$ are distinct. We clearly have, as in the classical case:

Proposition 4. If P is a hereditary radical then any ideal I of a ring R which has a P-primary representation has a proper representation.

Remark 6. As we noted in Example 2, when P is non-hereditary we cannot assume that the intersection of P-primary ideals A, B is P-primary even if $G(A) = G(B)$. Thus an ideal may have an irredundant P-primary representation but not one which is proper. Also note that ideals generally do not have P-primary representations even when P is hereditary. Example 1 gives a case in which $P = N$ is a hereditary radical and a ring R which, even more, is (left) Noetherian, but the 0 ideal is not the intersection of right P-primary ideals.

We suppose now that P is an arbitrary radical and that $I \lhd R$ has an irredundant P-primary representation. We will group by equal radicals; that is, we write $I = A_1 \cap \ldots \ldots \cap A_n$ where for some $t_j \geqq 1$ each $A_j = I_{1j} \cap \ldots \cap I_{t_j j}$, all I_{ij} are P-primary, and for each j there is a distinct $G_j = G(I_{ij})$ for all $i = 1, \ldots, t_j$. We will call the $\{A_j\}$ the *components* of the representation and the corresponding $\{G_j\}$ the *component radicals.*
For ideals A, B of a ring R write $(A:B) = \{x \in R \mid Bx \subseteqq A\}$.

Lemma. *Let A be a left P-primary ideal of a ring R. If $B = I_1 \cap \ldots \cap I_k$ for a set $\{I_i\}$ of ideals of R such that all $G(I_i) \subsetneqq G(A)$ then $(A:B) = A$.*

Proof. Clearly $A \subseteqq (A:B)$ and we have $I_1 I_2 \ldots I_k (A:B) \subseteqq A$. But $G(I_i) \subsetneqq G(A)$ implies $I_i \subsetneqq G(A)$ so we may remove I_1, and by continuing the argument we arrive at $(A:B) \subseteqq A$.

Theorem 3. *If an ideal I of an arbitrary ring has two irredundant P-primary representations with components $\{A_j\}$ $(j = 1, \ldots, n)$ and $\{B_j\}$ $(j = 1, \ldots, m)$ with $\{G_j\}$ and $\{H_j\}$ their respective component radicals, then $n = m$ and (possibly reordering) all $G_j = H_j$.*

PROOF. At least one of the $\{G_j, H_j\}$ is not properly contained in any of the others and we may suppose it is G_1. Then $G_1 \not\supseteq G_j$ for all $j \geqq 2$ and if G_1 is not equal to any H_j then $G_1 \not\supseteq H_j$ for all j. Now we are assuming the representations are left P-primary and so there are left P-primary $\{J_{ij}\}$ with $B_j = \cap J_{ij}$ and $H_j = G(J_{ij})$. By the Lemma we have $(J_{ij}: A_1) = J_{ij}$ for all j and $(I_{ij}: A_1) = I_{ij}$ for all $j \geqq 2$. Thus $(A_j: A_1) = \cap (I_{ij}: A_1) = A_j$ for $j \geqq 2$ and similarly $(B_j: A_1) = B_j$ for all j. Since $(A_1: A_1) = R$ it is clear that $(I: A_1) = A_2 \cap \ldots \cap A_n = B_1 \cap \ldots \cap B_m$ contradicting irredundance.

Thus G_1 equals one of the H_j and we may suppose $G_1 = H_1$. Thus $G_1 = H_1 \not\supseteq G_j, H_j$ for all $j \geqq 2$ and using the Lemma in the same way, we obtain

$$(I: A_1 \cap B_1) = \cap (A_j: A_1 \cap B_1) = (B_j: A_1 \cap B_1) = A_2 \cap \ldots \cap A_n = B_2 \cap \ldots \cap B_m,$$

and we are done by induction.

This is all simpler when (as in the case P is hereditary) the representations are proper. Thus:

Corollary 1. If an ideal I of R has two proper P-primary representations $I = I_1 \cap \ldots \cap I_n = J_1 \cap \ldots \cap J_m$ then $n = m$ and (possibly reordering) all $G(I_i) = G(J_i)$.

If $\{G_i\}$ is a finite set of ideals of a ring then a subset with the property that none of its members contain any of the G's outside the set is called *isolated*; that is (possibly relabeling), if there is some $k \geqq 1$ such that $G_j \not\supseteq G_i$ for all $j \leqq k$ and $i > k$. Note that we can certainly find at least one G_i which doesn't contain any other of the G's so there exists at least a one element isolated subset.

Theorem 4. *Let an ideal I of R have two P-primary representations $I = A_1 \cap \ldots \cap A_n = B_1 \cap \ldots \cap B_n$ with component radicals $G_i = H_i$. If $\{G_1, \ldots, G_k\}$ is an isolated subset of the $\{G_i\}$ then $A_1 \cap \ldots \cap A_k = B_1 \cap \ldots \cap B_k$.*

PROOF. Let $C = \bigcap_{j \leqq k} A_j$, $D = \bigcap_{i > k} A_i$, $E = \bigcap_{j \leqq k} B_j$, and $F = \bigcap_{i > k} B_i$. Then $I = C \cap D = E \cap F$ so $(I: D \cap F) = (C: D \cap F) = (E: D \cap F)$. But $G_i \not\supseteq G_j$ for all $i > k$ and $j \leqq k$ so it follows immediately from the Lemma that $(C: D \cap F) = C$ and $(E: D \cap F) = E$.

Corollary 2. Let an ideal I of R have two proper P-primary representations $I = I_1 \cap \ldots \cap I_n = J_1 \cap \ldots \cap J_n$ with all $G(I_i) = G(J_i)$. If k is a positive integer such that $G(I_j) \not\supseteq G(I_i)$ whenever $j \leqq k$ and $i > k$ then $I_1 \cap \ldots \cap I_k = J_1 \cap \ldots \cap J_k$.

4. Change of radical and the upper *P*-primary radicals

We will first consider P-primariness under certain changes of radical.

Proposition 5. If an ideal I of R is P-primary, then it is P'-primary for every radical $P' \supseteq P$.

PROOF. This is clear since $G(I)/I \in P \subseteq P'$ so $G(I)/I \subseteq P'(R/I)$ and thus $G(I) \subseteq G'(I)$.

Remark 7. Since the prime ideals are the 0-primary ideals, this yields the result (already noted) that prime ideals are P-primary for all radicals P.

Proposition 6. Let radicals $P \subseteqq P'$ and suppose $G(A)=G(B)$ for ideals A, B of R, then also $G'(A)=G'(B)$.

PROOF. Write $G=G(A)=G(B)$ and $G'=G'(A)$. We have $A \subseteqq G$ and $G'/A \in P'$ so $G'/G \in P'$. Also $G/B \in P \subseteqq P'$ so by the homomorphic extension property $G'/B \in P'$. Thus $G'=G'(A) \subseteqq G'(B)$ and by symmetry they are equal.

For class M of rings let $\mathcal{I}M$ denote the hereditary closure of M, namely $\mathcal{I}M=\{K | K=A_n \lhd A_{n-1} \lhd \ldots \lhd A_1=R \in M$ for some $n \geqq 1\}$. Then clearly $\mathcal{I}M$ is the smallest hereditary class containing M and it is well-known that $L\mathcal{I}M$ is the smallest hereditary radical containing M. Thus we have

Corollary 3. If A, B are P-primary ideals of R with $G(A)=G(B)$ then $A \cap B$ is P'-primary for $P'=L\mathcal{I}P$ where $G'(A \cap B)=G'(A)=G'(B)$.

Remark 8. It follows that whenever an ideal has a P-primary representation it also has a proper P'-primary representation for $P'=L\mathcal{I}P$. Note that while the $\{A_i\}$ of Theorem 3 would then be P'-primary the representation might still not be proper since $G_i \neq G_j$ need not imply that $G'(A_i) \neq G'(A_j)$.

Theorem 5. *For an arbitrary ideal I of a ring R there exists a radical P such that I is P-primary which is minimal in the sense that if I is already Q-primary for a hereditary radical Q then $P \subseteqq Q$.*

PROOF. Let $\{D_i\}$ be the set of all ideals of R which properly contain I and let $A=\sum_i l(D_i)$ where $l(D_i)=\{x \in R | xD_i \subseteqq I\}$. Define $P=L\{A/I\}$ and suppose $CD \subseteqq I$ for ideals C, D of R. If $D \subseteqq\!\!\!\!/\ I$ then $C \subseteqq l(D+I)$ where $D+I$ properly contains I. Thus $C \subseteqq A$ and so $C/I \subseteqq A/I \in P$. Therefore $C \subseteqq G(I)$ that is R is left P-primary. Now if I is left Q-primary then since $l(D_i)D_i \subseteqq I$ where $D_i \subseteqq\!\!\!\!/\ I$, it follows that $l(D_i)/I \subseteqq Q(R/I)$. Thus $A/I \subseteqq Q(R/I)$ and if Q is a hereditary radical then $A/I \in Q$ so that $P=L\{A/I\} \subseteqq Q$.

Remark 9. Note that there is a smallest hereditary radical P' relative to which the ideal I of the last theorem is left P'-primary, namely $P'=L\mathcal{I}P=L\mathcal{I}\{A/I\}$.

Proposition 7. Let P be an arbitrary radical and suppose $I=I_1 \cap \ldots \cap I_n$ where the $\{I_i\}$ are P-primary ideals of a ring R. Then I has a P'-primary representation for any $P' \supseteqq \{G(I_i)/I_i\}$.

PROOF. We have $G(I_i)/I_i \in P'$ so $G(I_i) \subseteqq G'(I_i)$. If I is left P-primary and $CD \subseteqq I_i$ with $D \subseteqq\!\!\!\!/\ I_i$ then $C \subseteqq G(I_i) \subseteqq G'(I_i)$. Thus I_i is left P'-primary.

Corollary 4. Every ideal of a commutative Noetherian ring has a P-primary representation for any radical $P \supseteqq B$.

PROOF. This follows from the classical result [1, p. 9] that every ideal of a commutative Noetherian ring has a "primary" representation.

We now show that if P is a hereditary radical then P-primariness is also a hereditary property, and (by an example) that this need not be true for non-hereditary P.

Theorem 6. *Let P be a hereditary radical and R a P-primary ring. Then any ideal I of R is also a P-primary ring.*

Proof. If $I\in P$ then it is automatically P-primary so let $I\notin P$. We suppose R is left P-primary and C, D ideals of I such that $CD=0$. Then $CID=0$ and $RCID=0$ so $(RC+C)ID=0$. If $ID\neq 0$ then by Proposition 1 we have $RC+C\subseteqq P(R)$ so $C\subseteqq P(R)\cap I=P(I)$. On the other hand, if $ID=0$ then $IRD=0$. Since $I\notin P$ so $I\nsubseteq P(R)$, there is some $x\in I$ with $x\notin P(R)$. But then Proposition 2 implies $D=0$.

Example 3. Let $P=\{R|R^2=R\}$ which is known to be a non-hereditary radical (the "idempotent" radical). Let I be any ring such that $I^2=0$ and imbed I as an ideal in a ring R with unit. Then $R\in P$ so R is P-primary but $I^2=0$ with $I\nsubseteq P(I)=0$, so I is not P-primary.

From Theorem 6 we have that when P is hereditary the three P-primary classes Q_l, Q_r, and $Q=Q_l\cap Q_r$ are all hereditary, so their upper radical exist. Since prime rings are P-primary we have inclusions

$$A\subseteqq Q=Q_l\cap Q_r\subseteqq Q_l(\text{or } Q_r)\subseteqq Q_l\cup Q_r,$$

so

$$U(Q_l\cup Q_r)\subseteqq UQ_l(\text{or } UQ_r)\subseteqq UQ\subseteqq UA=B.$$

We can ask when these radicals are distinct, and answer the question in part by:

Theorem 7. (1) *If* $P\cap B\neq 0$ *then* $UQ\neq B$, *and* (2) *If* $P\cap B=0$ *then* $U(Q_l\cup Q_r)=$ $=UQ_l=UQ_r=UQ=B$.

Proof. (1) If $0\neq R\in P\cap B$ then R is automatically P-primary. Thus $R\in Q$ so $R\notin UQ$ and hence $UQ\neq B$. (2) If $R\in B$ then every image $\bar{R}\in B$, and since B is a hereditary radical its ideals are all in B. But $P\cap B=0$ so that $P(\bar{R})=0$ and if $\bar{R}$ were left or right P-primary it would be prime, contradicting $\bar{R}\in B$. We conclude that $\bar{R}\notin(Q_l\cup Q_r)$, that is $R\in U(Q_l\cup Q_r)$.

Note that from part (2) of this theorem when $P\cap B=0$ the upper P-primary radicals exist (and are in fact hereditary radicals) whether or not P is itself hereditary. Thus by [4, Theorem 1, p. 219].

Corollary 5. If P is an arbitrary radical for which $P\cap B=0$ then: (1) If R is a P-primary ring then it has an image $0\neq\bar{R}$ each non-zero ideal of which contains a P-primary ideal, and (2) For an arbitrary ring R if R has a non-zero ideal containing a P-primary ideal then R itself contains a P-primary ideal.

Remark 10. (1) of Theorem 7 leaves open the possibility that in case $P\cap B\neq 0$ some or all of the upper P-primary radicals might be distinct. Note that the ring R of Example 1 which distinguishes between Q_l and Q_r does not separate UQ_l and UQ_r since it has a prime image $(R/(x)=F[y])$ and hence is in neither.

Remark 11. Throughout this paper we have been tacitly assuming that we are working in the category of all associative rings. However, the definitions of the P-primaries do not require associativity, and it may be of interest to examine how much of this theory remains when generalized to the not necessarily associative rings. Propositions 1 and 2 must, of course, be dropped, but Theorems 1 and 2 and the first four remarks remain valid.

Note that if P is a radical defined in the category of all not necessarily associative rings and W is the class of all associative rings then $\bar{P}=P\cap W$ is a radical in W. On the other hand any radical $\bar{P}$ in W has an extension P (the lower radical $P=L\bar{P}$

for L defined in the class of not necessarily associative rings) such that $\bar{P}=P\cap W$. Since an associative ring R is P-primary if and only if it is $\bar{P}$-primary the three examples may still be used (or nonassociative examples could no doubt be constructed).

For uniqueness, one would need to redefine $(A:B)=\Sigma T$ where $T\lhd R$ and $BT\subseteqq A$. The Lemma is now useless since it works only for case $k=1$ so for the proof of Theorem 3 we must use Theorem 2 rather than the Lemma. We must therefore limit ourselves to P a hereditary radical and Theorem 3 reads as does Corollary 1. Theorem 4 and its corollary are gone, but we can retain Propositions 4—7, Theorem 5, and the intervening remarks. Theorem 6 is, of course, gone so it is not clear whether or not in general the upper P-primary radicals exist. However, Theorem 7 is still valid so if $P\cap B=0$ they do exist (all equal to B) and Corollary 5 applies.

References

[1] RICHARD WIEGANDT, Radical and semisimple classes of rings, *Queens Papers in Pure and Applied Mathematics* (No. 37) *Queens University, Kingston Ontario* (1947).

[2] WOLFGANG KRULL, Idealtheorie, Ergebnisse der Mathematik und ihrer Grenzgebiete, V. 46 (1968).

[3] A. E. HOFFMAN and W. G. LEAVITT, A note on the termination of the lower radical construction, *J. London Math. Soc.*, **43** (1968).

[4] PAUL O. ENERSEN and W. G. LEAVITT, The upper radical construction, *Publ. Math. (Debrecen)* **20** (1973), 219—222.

(Received March 4, 1975)