# Investigations in the powersum theory V.

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To the memory of A. Kertész

**1.** In our paper [1] to appear we proved as a first step to a general theorem the following.

**Theorem A.** *Let us consider the equation*

(1.1) $$Y^{(n)}+a_{n-1}Y^{(n-1)}+\ldots+a_0=0,\quad Y=Y(t),\ a_\nu \text{ real constans.}$$

*We suppose that the equation*

(1.2) $$f(z)\doteq a_n z^n+a_{n-1}z^{n-1}+\ldots+a_0=0\quad (a_n=1)$$

*has no zeros in the strip*

(1.3) $$|\mathrm{Im}\, z|\leqq \Lambda$$

*Then all real solutions $Y(t)\not\equiv 0$ change sign in every real interval of length*

(1.4) $$>\frac{n\pi}{2\Lambda}.$$

*This is bestpossible for all $n\geqq 2$ and $\Lambda>0$.*

An analogous result could have been proved for the difference equation

(1.5) $$x_{n+l}+b_{n-1}x_{n+l-1}+\ldots+b_1x_{l+1}+b_0x_l=0,\qquad l=0,1,\ldots$$

**2.** The proof was based on a "onesided" powersum theorem (see [2]) which proved to be useful in several situations e.g. in analytical number theory. We shall not formulate it here but mention that in its proof the following auxiliary problem played an essential role. Let $\pi_n(x)$ be a polynomial of $n^{th}$ degree with $\pi_n(0)=1$ say so that all of its roots are outside the angle

(2.1) $$|\mathrm{arc}\, z|\geqq \varkappa$$

with a $0<\varkappa\leqq\frac{\pi}{2}$. One has to multiplity it by a polynomial $q_k(x)$ with $g_k(0)=1$ and possibly small degree $k$ so that all coefficients of $\pi_n(x)q_k(x)$ are nonnegative. In order to get (1.4) we had to solve in our paper IV. the corresponding problem when in addition the restriction

$$\text{the coefficients of } \pi_n(x) \text{ are real} \tag{2.2}$$

is satisfied. In the present paper — which is strictly speaking a digression — we are going to show that a "theorem of alternative" equivalent to the Farkas theorem from the theory of linear inequalities gives possibility to go from the abovementioned "multiplication problem" over *directly* to the result (1.5) and then after a simple passage to limit to theorem A. Calling a vector $X$ in $R^n$ positive, $X>0$ when all coordinates are positive the theorem of alternative in question* asserts that if $G$ is a matrix with real elements then exactly one of the inequalities

$$GX \geqq 0, \quad GX \not\equiv 0 \quad (X \text{ column vector}) \tag{2.3}$$

$$YG = 0, \quad Y > 0 \quad (Y \text{ row vector}) \tag{2.4}$$

is solvable.

**3.** In order to prove theorem A on this way we shall make first a detour and consider the product $\varphi(z)q_k(z)$ where

$$\begin{aligned} \varphi(z) &= d_0+d_1z+\dots+d_nz^n, \\ g_k(z) &= x_0+x_1z+\dots+x_kz^k, \end{aligned} \tag{3.1}$$

and the $x_\nu's$ are real variables. The coefficients of the product, if $X=(x_0, x_1, \dots, x_k)$, form a columnvector which can be written as

$$DX \tag{3.2}$$

where $D$ is an $(n+k+1)\times(k+1)$ matrix whose explicit form for $k\leqq n$ resp. $k>n$ is

$$D = \begin{pmatrix} d_0 & 0 & \cdots & 0 \\ d_1 & d_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ d_k & d_{k-1} & \cdots & d_0 \\ d_{k+1} & d_k & \cdots & d_1 \\ \vdots & & & \vdots \\ d_n & d_{n-1} & \cdots & d_{n-k} \\ 0 & d_n & \cdots & d_{n-k+1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

*) See J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, *Berlin*, 1970, p. 23.

resp.

$$D = \begin{pmatrix} d_0 & 0 & & \cdots & & & & & 0 \\ d_1 & d_0 & 0 & \cdots & & & & & 0 \\ \vdots & & & \ddots & & & & & \vdots \\ d_n & d_{n-1} & & \cdots & d_0 & 0 & & \cdots & 0 \\ 0 & d_n & & \cdots & d_1 & d_0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots & \vdots \\ 0 & & & & d_n & d_{n-1} & & \cdots & d_0 \\ \vdots & & & & & d_n & & \ddots & \vdots \\ 0 & \cdots & & & & & & 0 & d_n \end{pmatrix}$$

We define $k_0 = k_0(\varphi)$ as the minimum of integer $k$'s for which there is a suitable $q_k^*(z) \not\equiv 0$ so that all coefficients of the product are nonnegative; if there is no such $k$ (when $\varphi(z)$ has a positive zero of odd multiplicity e.g.) then let $k_0 = \infty$. We may suppose $k_0 \geqq 1$. Choosing $k$ in (3.1) as $(k_0 - 1)$ the system

$$DX \geqq 0, \quad DX \not\equiv 0 \tag{3.3}$$

is *not* solvable. But then from the above quoted theorem of alternative the system

$$YD = 0, \quad Y > 0 \tag{3.4}$$

is solvable. If

$$Y = (y_0, y_1, \ldots, y_{n+k_0}) \tag{3.5}$$

is such a solution then we have

$$y_\nu > 0 \qquad (\nu = 0, 1, \ldots, (n+k_0)) \tag{3.6}$$

and the relations

$$\begin{aligned} d_0 y_0 + d_1 y_1 \quad &+ \ldots + d_n y_n &= 0 \\ d_0 y_1 + d_1 y_2 \quad &+ \ldots + d_n y_{n+1} &= 0 \\ \vdots \qquad & & \vdots \\ d_0 y_{k_0} + d_1 y_{k_0+1} &+ \ldots + d_n y_{n+k_0} &= 0 \end{aligned}$$

This means owing to $a_n = 1$ that the difference equation

$$d_n y_{m+n} + d_{n-1} y_{m+n-1} + \ldots + d_0 y_m = 0 \tag{3.7}$$

$$m \geqq 0$$

has a solution which is positive for

$$0 \leqq m \leqq k_0 + n. \tag{3.8}$$

Hence if $m_0$ stands for the maximal length of an interval where a solution of the equation (3.7) can be positive then we got

$$m_0 \geqq k_0+n. \tag{3.9}$$

**4.** Next we start with a solution $y_m^*$ of the equation (3.7) which is positive for an interval of length $m_0$. This means that the system

$$\begin{aligned} d_0y_0^*+d_1y_1^*+\dots+d_ny_n^* &= 0\\ d_0y_1^*+d_1y_2^*+\dots+d_ny_{n+1}^* &= 0\\ \dots\quad &\cdot \\ d_0y_{m_0-n}^*+d_1y_{m_0-n+1}^*+\dots+d_ny_{m_0}^* &= 0 \end{aligned} \tag{4.1}$$

has a solution with

$$y_j^* > 0 \qquad (j = 0, 1, \dots, m_0). \tag{4.2}$$

But this means that the system

$$YD_1 = 0, \quad Y > 0, \quad Y \text{ row vector}$$

is solvable by $Y=Y^*$ where $D_1$ stands for the $(m_0-n+1)\times(m_0+1)$ matrix

$$D_1 = \begin{pmatrix} d_0 & 0 & 0 & \dots & 0 \\ d_1 & d_0 & & \dots & 0 \\ \vdots & & & & \vdots \\ d_n & d_{n-1} & & \dots & \\ 0 & d_n & & \dots & d_0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \dots & 0\, d_n \end{pmatrix} \tag{4.3}$$

The application of the theorem of alternative gives then the *non* solvability of the system

$$D_1X \geqq 0, \quad D_1X \not\equiv 0.$$

Hence if $k_0$ has the previous meaning then

$$k_0+1 \geqq m_0-n+1. \tag{4.4}$$

This and (3.9) give the

**Theorem I.** *For the above defined $m_0$ and $k_0$ the relation*

$$m_0 = k_0+n$$

*holds.*

This theorem reduces the determination of maximal length of a positivity-interval of an arbitrary solution of the difference equation (3.7) to the determination of the minimal $k=k_0$ for which a $q_k(z)\not\equiv 0$ polynomial exists so that the product in (3.1) has nonnegative coefficients. Since the following theorem of alternative also holds

$$(4.5) \qquad GX > 0$$

resp

$$(4.6) \qquad YG = 0, \quad Y \geqq 0, \quad Y \neq 0$$

the repetition of the previous reasoning gives

**Theorem II.** *The relation*

$$m_1 = k_1 + n \quad Y \neq 0$$

*holds if $m_1$ stands for the maximal length of nonnegativity interval of an arbitrary solution of the difference equation* (3.7) *and $k_1$ is the minimal $k$ for which a $q_k(z)$ polynomial exists so that the product in* (3.1) *has positive coefficients.*

Obviously if we know upper bounds for $k_0 = k_0(\varphi)$ and $k_1 = k_1(\varphi)$ then we get upper bounds for $m_0$ resp. $m_1$.

**5.** It was shown in [2] in the important case

$$(5.1) \qquad \varphi^*(z) = r^2 - 2\cos\alpha\, rz + z^2$$

$$r > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

the inequality

$$k_0(\varphi^*) \leqq \left[\frac{\pi}{|\alpha|}\right] - 1$$

and in [3] where for the first time the general theory of inequalities was applied to that sort of equations that

$$(5.2) \qquad k_0(\varphi^*) = \begin{cases} \dfrac{\pi}{|\alpha|} - 2 & \text{if } \dfrac{\pi}{|\alpha|} \text{ is an integer,} \\ \left[\dfrac{\pi}{|\alpha|}\right] - 1 & \text{if } \dfrac{\pi}{|\alpha|} \neq \text{integer.} \end{cases}$$

In the general case (1.2) $f(z)$ is a product of factors of form

$$z + r_\nu \quad r_\nu > 0$$

and also of the form

$$(5.3) \qquad z^2 - 2\cos\alpha_\nu r_\nu z + r_\nu^2$$

$$r_\nu > 0, \quad 0 < |\alpha_\nu| \leqq \pi.$$

We have to care only with factors from (5.3) with

$$0 < |\alpha_\nu| < \frac{\pi}{2};$$

applying (5.2) to each them we get the general inequality

$$k_0(\varphi) \leqq \sum{}' \left(\frac{\pi}{|\alpha_\nu|}-2\right)+\sum{}'' \left(\left[\frac{\pi}{|\alpha_\nu|}\right]-1\right) \tag{5.4}$$

where $\sum'$ resp. $\sum''$ is the extended to the $\alpha_\nu$'s with

$$\frac{\pi}{|\alpha_\nu|} = \text{integer} \geqq 3,$$

resp.

$$\frac{\pi}{|\alpha_\nu|} > 2 \quad \text{and} \neq \text{integer}.$$

Suppose now that for $\varphi(z)$ with real coefficients there is a $\varkappa$ with

$$0 < \varkappa < \frac{\pi}{2} \tag{5.5}$$

so that $\varphi(z) \neq 0$ for

$$|\text{arc } z| \leqq \varkappa \tag{5.6}$$

Then (5.4) and (5.6) give at once

**Theorem III.** *If the polynomial $\varphi(z)$ of $n^{th}$ degree and real coefficients satisfies* (5.6) *then the inequality*

$$k_0(\varphi) \leqq \left[\frac{n}{2}\right]\left(\left[\frac{\pi}{\varkappa}\right]-1\right) \leqq \frac{n}{2}\left(\left[\frac{\pi}{\varkappa}\right]-1\right)$$

*holds.*

Combining it with theorem I. we get

**Theorem IV.** *If*

$$\varphi(z) = d_0+d_1z+\ldots+d_nz_n \quad (d_n = 1) \tag{5.7}$$

*with real $d_\nu$'s and* (5.6) *is satisfied then no solution $y_l$ of the difference equation*

$$d_0y_l+d_1y_{l+1}+\ldots+d_ny_{l+n} = 0 \tag{5.8}$$

*can be positive for more than*

$$\frac{n}{2}\left(\left[\frac{\pi}{\varkappa}\right]+1\right) \tag{5.9}$$

*consecutive integer l-values.*

**6.** Next we show that equality in (5.9) can be attained for each $0<\varkappa<\frac{\pi}{2}$ and even $n$. Let namely $n=2N$ and the coefficients $d_\nu^*$ in (5.7) be defined by

$$d_0^*+d_1^*z+\ldots+d_{2N}z^{2N} = (1-2\cos\alpha z+z^2)^N, \tag{6.1}$$

where $0<\alpha<\frac{\pi}{2}$ is such that $\frac{\pi}{\alpha}$ is not an integer (an unessential restriction). Then we have

$$\varkappa = \alpha$$

of course. Then all sequences $y_l^{**}$ of the form

$$y_l^{**} = (c_0+c_1 l+\ldots+c_{N-1}l^{N-1}) \sin(\alpha l+\vartheta) \tag{6.2}$$

$\vartheta$, $c_\nu$ real constants, $l$ variable satisfy the equation

$$d_0^* y_l+d_1^* y_{l+1}+\ldots+d_{2N}^* y_{l+2N} = 0 \tag{6.3}$$

Let $\vartheta$ be any number with

$$0 < \vartheta < \pi-\alpha\left[\frac{\pi}{\alpha}\right]$$

for which in addition none of the numbers

$$\frac{\nu\pi-\vartheta}{\alpha}, \quad \nu = 0, \pm 1, \ldots$$

are integers. Fixing such a $\vartheta=\vartheta_0$ and writting

$$\frac{\nu\pi-\vartheta_0}{\alpha} = \beta_\nu$$

we see at once that the sequence

$$y_l^{**} = \prod_{\nu=1}^{N-1} (\beta_\nu-l) \sin(\alpha l+\vartheta_0)$$

is of form (6.2) and

$$y_l^{**} > 0 \quad \text{for} \quad l = 0, 1, \ldots, N\left(\left[\frac{\pi}{\alpha}\right]+1\right)-1$$

indeed.

**7.** In order to deduce theorem A from theorem IV. we reformulate first theorem IV. If $z_1, z_2, \ldots, z_k$ are different zeros of the equation (5.7) then all solutions of equation (5.8) have the form

$$y_l = \sum_{\mu=1}^{k} Q_\mu(l) z_\mu^l \tag{7.1}$$

Where complex $z_\mu$'s occur only in conjugate pairs further the $Q_\mu(l)$'s are polynomials in $l$ so that

$$\sum_{\mu=1}^{k} \operatorname{degr} Q_\mu(x) = n-k \tag{7.2}$$

and $Q_\mu(x)$'s belonging to complex-conjugate $z_\mu$ have complex-conjugate coefficients and conversely, each such sequence (7.1)—(7.2) satisfies an (essentially uniquely determined) equation (5.8). Thus theorem IV. means that all sequences of type (7.1)—(7.2) with no $z_j$'s in the angle

$$|\mathrm{arc}\, z| \leqq \varkappa \tag{7.3}$$

are such that no consecutive $L$ terms can be positive if only

$$L > \frac{n}{2}\left(\left[\frac{\pi}{\varkappa}\right]+1\right). \tag{7.4}$$

**8.** Now we deduce quickly theorem A. Let $y$ be an arbitrary solution of (1.1) where (1.2)—(1.3) satisfied. Denoting the zeros of (1.2) by $\xi_j$ $(j=1, 2, \ldots, n)$ we have

$$Y(t) = \sum_{\mu=1}^{k} P_\mu(t) e^{\xi_\mu t} \tag{8.1}$$

where $\xi_1, \ldots, \xi_\mu$ are all pairwise different zeros of (1.2), the $P_\mu(t)$'s are polynomials with

$$\sum_{\mu=1}^{k} \mathrm{degr}\, P_\mu(t) = n-k. \tag{8.2}$$

The restrictions made imply that $Y(t)$ is real for real $t$. If $a$ is an arbitrary real number and $h$ is sufficiently small, both fixed then let us consider the sequence

$$y(l) \overset{\mathrm{def}}{=} Y(a+hl) = \sum_{\mu=1}^{k} P_\mu(a+hl) e^{\xi_\mu a} (e^{\xi_n h})^l. \tag{8.3}$$

Suppose now that for $\mu=1, 2, \ldots, n$

$$\min_{\mu} \; |\mathrm{Im}\, \xi_\mu| = \Lambda > 0 \tag{8.4}$$

and we apply theorem IV. as reformulated in *7.* with

$$z_\mu = e^{\xi_\mu h}, \quad Q_\mu(l) = P_\mu(a+hl) \qquad \mu = 1, 2, \ldots, k. \tag{8.5}$$

Then if $h$ is sufficiently small then we can take as $\varkappa$ in (7.3)

$$\varkappa = h\Lambda. \tag{8.6}$$

Hence owing to (7.4) there are integers $l_1$ and $l_2$ with

$$0 \leqq l_1 \neq l_2 \leqq L_1$$

and with

$$L_1 = \frac{n}{2}\left(\frac{\pi}{h\Lambda}+1\right)+c, \quad c > 0$$

so that

$$y(l_1) > 0, \quad y(l_2) \leqq 0.$$

Since

$$a \leqq a+hl_1 \leqq a+\frac{n}{2}\left(\frac{\pi}{\Lambda}+h\right)+ch,$$

$$a \leqq a+hl_2 \leqq a+\frac{n}{2}\left(\frac{\pi}{\Lambda}+h\right)+ch,$$

this means that

$$\max_{t\in I} Y(t) > 0 \quad \min_{t\in I} Y(t) \leqq 0$$

where

$$I=\left[a,\, a+\frac{n\pi}{2\Lambda}+\left(\frac{n}{2}+c\right)h\right].$$

Thus for $h\to 0$ we obtained again theorem A.

## References


[1] S. Dancs and P. Turán, Investigation in the power sum theory IV, Publ. Math. (Debrecen) **22** (1975), 123-131.

[2] P. Turán, On an improvement of some new onesided theorems, Acta Math. Hung. **11** (1960), 299-316.

[3] S. Dancs, Remarks on a paper of P. Turán, *Annales Univ. Scient. Budapestiensis de R. Eötvös Nominatae*, VII (1964), 133-141.


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