On $p^{\omega+n}$-projective abelian p-groups

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In memoriam Andor Kertész

Among the abelian p-groups without elements of infinite heights, only the direct sums of cyclic p-groups and the torsion-complete p-groups have satisfactory structure theories. In addition, a great deal is known of the direct sums of torsion-complete p-groups (see e.g. [2], section 73) and of the $p^{\omega+1}$-projective p-groups (see FUCHS and IRWIN [3]); in both cases, the socles—if viewed as vector spaces with valuation—determine the group structures.

The purpose of this note is to pursue the idea developed in [3] and to extend the results of [3] to the class of $p^{\omega+n}$-projective p-groups.

Recall that a $p^{\omega+n}$-*projective* p-group (where ω stands for the first infinite ordinal and n is a positive integer) can be defined in one of the following equivalent ways (cf. NUNKE [5], BENABDALLAH, IRWIN and RAFIQ [1]): it is a p-group A satisfying

(a) $p^{\omega+n}\,\mathrm{Ext}\,(A, G)=0$ for all groups G;

(b) it contains a p^n-bounded subgroup P (i.e. $p^nP=0$) such that A/P is a direct sum of cyclic groups;

(c) it is isomorphic to F/U for some direct sum F of cyclic p-groups and a p^n-bounded subgroup U of F.

Our present approach is similar to the one developed in [3]: we shall view the p^n-*socle* $A[p^n]=\{a\in A \mid p^na=0\}$ of A as an abelian group with valuation where the values are given by the height function $h(a)$. The main result states that two $p^{\omega+n}$-projective p-groups are isomorphic exactly if their p^n-socles are isometric as valued abelian groups (Theorem 5).

We have found for this theorem a more direct proof than the one used in [3] to prove the analogous theorem. In the present case, however, we have been unable to get the same amount of information about the structures of $p^{\omega+n}$-projectives. As a matter of fact, several questions settled in the case of $p^{\omega+1}$-projectives have been left open, of which the most interesting is John Irwin's conjecture on the ubiquity of $p^{\omega+n}$-projectives: does every p-group which is not $p^{\omega+n-1}$-projective contain a *proper* $p^{\omega+n}$-projective p-group (i.e. one which fails to be $p^{\omega+n-1}$-projective)?

By a group we shall mean throughout an abelian p-group A where p is a prime. For the basic definitions and fundamental results we refer to our book [2]. As usual, $p^\sigma A$ is defined for all ordinals σ by setting $p^{\sigma+1}A=p(p^\sigma A)$ and $p^\varrho A=\bigcap_{\sigma<\varrho} p^\sigma A$ whenever

ϱ is a limit ordinal. For the sake of simplicity, we assume A reduced, i.e. $p^\tau A=0$ for some ordinal τ. By the *height* $h(a)$ of an element $a\neq 0$ in A is meant the ordinal σ if $a\in p^\sigma A\backslash p^{\sigma+1}A$. We set $h(0)=\infty$.

A subgroup N of A is said to be *nice* if $p^\sigma(A/N)=(p^\sigma A+N)/N$ for all ordinals σ. I.e. every coset of A mod N can be represented by an element $a\in A$ of the same height.

If P is a p^n-bounded subgroup of A and G is an arbitrary subgroup with $G[p^n]=P$, we then say G is *supported* by P. In particular, $p^\sigma A$ is supported by $p^\sigma A[p^n]$.

1. Let A be a p-group. By a *valuation* of A we shall mean a function $v\colon A\to\Gamma\cup\{\infty\}$ (where Γ denotes the class of ordinals and the symbol ∞ is regarded as larger than any ordinal) such that

(i) $v(a)=\infty$ if and only if $a=0$;

(ii) $v(ma)=v(a)$ or $>v(a)$ according as the integer m is prime to p or not;

(iii) $v(a+b)\geqq\min(v(a), v(b))$ for all $a, b\in A$.

Two groups with valuations are called *isometric* if there is a value-preserving isomorphism between them. A *morphism* between two valued groups is a group theoretical homomorphism which does not decrease values.

Let C be a subgroup of a group A with valuation v. Then the restriction of v to C makes C into a group with valuation. The quotient A/C carries the induced valuation: $v(a+C)=\sup_{c\in C} v(a+c)$ or $=h(a+C)=$ the height of the coset $a+C$ in the quotient A/C whichever is larger, and the canonical map $A\to A/C$ is a morphism of valued groups. A reduced p-group A can always be regarded as being equipped with the valuation $h=$height function, or if it is a subgroup of a reduced p-group G, it can be furnished with the valuation inherited from G.

A cyclic group $\langle a\rangle$ of order p^n is said to be *fundamental* if its valuation is given by

$$v(x) = v(a)+h(x)\quad (x\in\langle a\rangle)$$

where h denotes the height function in $\langle a\rangle$. More generally, a p-group A with valuation v will be called *σ-fundamental* ($\sigma\in\Gamma$) if in the subgroup $A^\sigma=\{a\in A|v(a)\geqq\sigma\}$ the valuation is given by $\sigma+h(a)$ with h computed in A^σ. (A cyclic p-group $\langle a\rangle$ is thus fundamental exactly if it is $v(a)$-fundamental.)

Let A_i ($i\in I$) be p-groups with valuations. By their direct sum $A=\oplus A_i$ is meant their group theoretical direct sum equipped with the valuation

$$v(\sum a_i) = \min_i v(a_i)\quad (a_i\in A_i).$$

In particular, if all A_i are fundamental cyclic groups $\langle a_i\rangle$ of order p^n, with $v(a_i)=\alpha_i$, then their direct sum A will be free in the sense that any function f from the basis $\{a_i\}_{i\in I}$ into a p^n-bounded valued group G such that $v(a_i)\leqq v(f(a_i))$ extends to a unique morphism $A\to G$ (as valued groups).

2. Our study starts with the characterization of those p^n-bounded groups P with valuation which can appear as subgroups in $p^{\omega+n}$-projective p-groups A such that A/P is a direct sum of cyclics.

Theorem 1. *For a p^n-bounded group P with valuation to be a subgroup of a $p^{\omega+n}$-projective p-group A such that A/P is a direct sum of cyclic groups, it is necessary and sufficient that*

(1) *the non-zero elements in P have values $<\omega+n$;*

(2) *P is ω-fundamental.*

Let A be a $p^{\omega+n}$-projective p-group and P a p^n-bounded subgroup of A such that A/P is a direct sum of cyclics. Then the subgroup $p^\omega A$ of elements of infinite height is contained in P, hence $p^{\omega+n}A=0$ and (1) follows. Since the heights of elements in $p^\omega A$ satisfy (2), the proof of necessity is complete.

Conversely, let P be a p^n-bounded group with valuation satisfying (1) and (2). For every $x\in P$ whose value is an integer n_x, select a fundamental cyclic group $\langle b_x\rangle$ of order p^n with $v(b_x)=n_x$, and for every $y\in P$ whose value is ω, choose infinitely many fundamental cyclic groups $\langle b_y^1\rangle, \dots, \langle b_y^k\rangle, \dots$ of order p^n with $v(b_y^k)=k$. The direct sum $B=\bigoplus_x\langle b_x\rangle\oplus\bigoplus_{y,k}\langle b_y^k\rangle$ is then a free, valued p^n-bounded group such that the function $b_x\to x$, $b_y^k\to y$ extends to a morphism $\varphi\colon B\to P$, and the natural map between $B/\mathrm{Ker}\,\varphi$ and P is an isometry; in fact, this follows at once from (1) and (2). For every b_x, b_y^k, select cyclic groups $\langle a_x\rangle, \langle a_y^k\rangle$ of orders p^{n+n_x} and p^{n+k}, respectively. Then under the correspondence $b_x\to p^{n_x}a_x$, $b_y^k\to p^k a_y^k$, B can be identified with a subgroup of $F=\bigoplus_x\langle a_x\rangle\oplus\bigoplus_{y,k}\langle a_y^k\rangle$ such that the heights of elements of B in F are given by the above valuation of B. Then $A=F/\mathrm{Ker}\,\varphi$ will be $p^{\omega+n}$-projective whose quotient mod $B/\mathrm{Ker}\,\varphi$ is a direct sum of cyclic groups.

3. Let A be a $p^{\omega+n}$-projective p-group and P a p^n-bounded subgroup of A such that A/P is a direct sum of cyclic groups. This P is not uniquely determined. But if we have another p^n-bounded P' in A with A/P' a direct sum of cyclics, then $P\cap P'$ has also the property that $A/(P\cap P')$ is a direct sum of cyclics (since it is a subgroup of $(A/P)\oplus(A/P')$). In order to learn a bit more about these P, we introduce the following concept.

Let S be a p-group with valuation. We will call S *distinctive* if there is a monomorphism of S into a direct sum of cyclic p-groups which does not decrease valuation. Keeping the notations of the preceding paragraph, we see that both $P/(P\cap P')$ and $P'/(P\cap P')$ are distinctive.

The following theorem records the most relevant fact we need in our discussion of distinctive p-groups.

Theorem 2. *Let G be a p-group and S a subgroup of G such that G/S is a direct sum of cyclic groups. Let S be equipped with the valuation given by the heights in G. If S is distinctive, then G is a direct sum of cyclic groups.*

Let $\psi\colon S\to C$ be a homomorphism of S into a direct sum of cyclics C which does not decrease heights. There is a group H and a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & S & \to & G & \to & G/S & \to & 0 \\ & & \downarrow\psi & & \downarrow\psi^* & & \| & & \\ 0 & \to & C & \to & H & \to & G/S & \to & 0 \end{array}$$

where $\psi^*\colon G\to H$ is monic (ψ being monic). It is easy to check that the second row is pure-exact. Since G/S is pure-projective ($=p^\omega$-projective), the second row splits. Thus H is a direct sum of cyclics, and so are its subgroups. Consequently, G is a direct sum of cyclics, indeed.

By making use of a corollary to a rather deep theorem by P. HILL (see e.g. [2], Theorem 81.4), we can give another proof. Since S is closed in the p-adic topology of G, it is nice in G, and since G/S is totally projective, the cited theorem guarantees that ψ extends to a homomorphism $\varphi\colon G\to C$. This together with the natural homomorphism $G\to G/S$ yields an embedding $G\to C\oplus G/S$, a direct sum of cyclics.

As an immediate corollary, we can state:

Corollary 1. *Suppose A is a $p^{\omega+n}$-projective p-group and P is a p^n-bounded subgroup with A/P a direct sum of cyclics. If P_0 is a subgroup of P such that P/P_0 is distinctive, then A/P_0 is again a direct sum of cyclic groups.*

Using the same idea, we can answer the question as to when a $p^{\omega+n}$-projective p-group is properly $p^{\omega+n}$-projective.

Theorem 3. *Let A be a $p^{\omega+n}$-projective p-group and P a p^n-bounded subgroup of A such that A/P is a direct sum of cyclic groups. A is $p^{\omega+n-1}$-projective if and only if P contains a p^{n-1}-bounded subgroup P' such that P/P' is distinctive.*

Sufficiency is a simple consequence of Corollary 1. To prove necessity, let Q be a p^{n-1}-bounded subgroup of A with A/Q a direct sum of cyclics. Then P/P' with $P'=P\cap Q$ is evidently distinctive.

4. We turn our attention to the structure of $p^{\omega+n}$-projective p-groups. Recall that the σth relative invariant $f_\sigma(G, A)$ of a subgroup G of a p-group A is defined as the dimension of the quotient

$$p^\sigma A[p]/((p^{\sigma+1}A+G)\cap p^\sigma A[p]),$$

viewed as a vector space over the integers mod p (cf. Hill [4]; see also [2]).

The following lemma is fundamental.

Lemma. *Let G be a p^n-bounded subgroup of a p-group A. Then the relative invariants $f_\sigma(G, A)$ of G in A can be computed by using only the elements of $A[p^n]$.*

We can rewrite $(p^{\sigma+1}A+G)\cap p^\sigma A[p]$ as $(p^{\sigma+1}A+G)[p]\cap p^\sigma A[p]$. Our assertion will follow at once if we can show that

$$(p^{\sigma+1}A+G)[p] = (p^{\sigma+1}A[p^n]+G)[p].$$

But if $p(a+g)=0$ where $a\in p^{\sigma+1}A$ and $g\in G$, then $p^n a=-p^n g\in p^n G=0$ implies $a\in p^{\sigma+1}A[p^n]$.

It is now easy to prove the next result which is stated in a more general fashion than needed for our purposes.

Theorem 4. *Let A and C be p-groups. Suppose that*

(i) *there are p^n-bounded nice subgroups, P and Q, in A and C, respectively, such that A/P and C/Q are totally projective (or, moreover, direct sums of cyclics);*

(ii) *there is a height-preserving isomorphism*

$$\varphi\colon A[p^n]\to C[p^n]$$

such that $\varphi P=Q$.

Then $A\cong C$.

Since P, Q are p^n-bounded and φ is height-preserving, from Lemma it follows at once that the relative invariants of P in A are equal to those of Q in C. Therefore, we have a height-preserving isomorphism $\varphi|P$ between the nice subgroups P and Q with totally projective quotients A/P, C/Q and the same relative invariants. The existence of an isomorphism $A \to C$ (extending $\varphi|P$) follows at once from Hill's theorem [4] (cf. also [2, Theorem 83.4]).

Our main objective is to improve on Theorem 4 in the special case when A/P and C/Q are direct sums of cyclic groups. Observe that in this case P and Q are automatically nice subgroups (for subgroups closed in the p-adic topology are always nice). We want to show that even the hypothesis $\varphi P = Q$ can be removed.

Theorem 5. *Two $p^{\omega+n}$-projective p-groups, A and C, are isomorphic precisely if there is a height-preserving isomorphism between $A[p^n]$ and $C[p^n]$.*

The necessity being obvious, suppose $\varphi: A[p^n] \to C[p^n]$ is a height-preserving isomorphism. Since A and C are $p^{\omega+n}$-projective p-groups, there exist p^n-bounded subgroups P in A and Q in C such that A/P and C/Q are direct sums of cyclic p-groups. Consider the quotient group $A/(P \cap \varphi^{-1}Q)$ qua an extension of $P/(P \cap \varphi^{-1}Q)$ by A/P. The map

$$x + (P \cap \varphi^{-1}Q) \mapsto \varphi x + Q \quad (x \in P)$$

of $P/(P \cap \varphi^{-1}Q)$ into C/Q is monic, and it evidently does not decrease heights. Application of Theorem 2 leads us to conclude that $A/(P \cap \varphi^{-1}Q)$ is again a direct sum of cyclic groups. For reasons of symmetry, $C/(\varphi P \cap Q)$ is likewise a direct sum of cyclic groups. We are now in the situation of Theorem 4, with $P \cap \varphi^{-1}Q$ and $\varphi P \cap Q$ playing the roles of P and Q, respectively. Consequently, A and C are isomorphic, as claimed.

In other words, Theorem 5 states that $p^{\omega+n}$-projective p-groups are completely characterized by their p^n-socles if regarded as groups with valuations.

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