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Optimal solutions of an alternative Cauchy equation

By DANIELA RUSCONI (Milano)

Abstract. We consider the alternative Cauchy functional equation

$$f(x+y)-f(x)-f(y)\neq 0 \quad \text{implies} \quad g(x+y)-g(x)-g(y)=0.$$

A characterization of new classes of solutions is given, when f and g are real functions.

1. Introduction

In the last years the alternative Cauchy equation

$$f(x+y)-f(x)-f(y)\neq 0 \quad \text{implies} \quad g(x+y)-g(x)-g(y)=0, \tag{1}$$

where f,g are unknown functions from a group $(X,+)$ into a group $(S,+)$, has been extensively studied (see [1], [2], for a rich bibliography). Among the results concerning the previous equation, some allow to write the local or global solutions of (1) when $(X,+)=(\mathbb{R},+)$ and g (or f) satisfies a suitable topological condition ([3]).

In this paper we suppose $(X,+)=(\mathbb{R},+)$ and we give some conditions which permit to extend a local solution of (1) and to characterize the solutions of (1) when g (or f) satisfies a weak topological hypothesis.

2. Notations and preliminaries

Denote by $\mathbb{Z}$ and $\mathbb{N}$ the classes of the integers and non-negative integers respectively, and by $p_i:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $i=1,2,3$, the maps given by

$$p_1(x,y)=x, \quad p_2(x,y)=y, \quad p_3(x,y)=x+y.$$

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Given a subset $E \subseteq \mathbb{R}\times\mathbb{R}$ and a function $g : p_1(E)\cup p_2(E)\cup p_3(E) \to S$ we define

$$\Omega_g = \{(x, y) \in E : g(x + y) \neq g(x) + g(y)\},$$
$$A_g = \{(x, y) \in E : g(x + y) = g(x) + g(y)\},$$

and we denote by Ω_g^0 and A_g^0 the interior of Ω_g and A_g respectively.

Let V be an open real interval; a function $g : V \to S$ is said *locally affine in* $x \in V$ if there exists $a \in \mathrm{Hom}(\mathbb{R}, S)$ such that $g(x + u) = g(x) + a(u)$ for all u in an open interval $U \ni 0$; g is said *locally affine in* V if it is locally affine in each point of V and in this case ([3] prg. 2) we have $g(x) = a(x) + \alpha$ for some constant α.

Denote by J a real open interval of the form $(0, a)$, $0 < a \leq +\infty$, and let $g : J \to S$, $f : J \to$S. We introduce the following notations:

$$\xi_t(x) = g(x) + g(t - x), \quad \varphi_t(x) = f(x) + f(t - x),$$
$$t \in 2J,\ x \in J \cap (t - J);$$
$$\Delta_\tau g(x) = g(x + \tau) - g(x), \qquad \tau \in J, \quad x \in (0, a - \tau);$$
$$H_t = H_t(J) = \{x \in (0, a - t) : g(x + t) = g(x) + g(t)\}, \quad t \in J;$$
$$r_t = \{(x, y) : x + y = t\},\ t \in \mathbb{R};$$
$$T_t = \{(x, y) : x, y, x + y \in (0, t)\} \quad \text{and} \quad Q_t = \{(0, t) \times (0, t)\}, \quad t > 0\ .$$

Obviously it is $\xi_t(x) = \xi_t(t - x)$ and $\varphi_t(x) = \varphi_t(t - x)$.

A pair (f, g) is a solution of (1) in $E \subseteq \mathbb{R}\times\mathbb{R}$ if $p_1(E)\cup p_2(E)\cup p_3(E)$ is contained in the domain of f and g and (1) holds for every $(x, y) \in$E.

Let (f, g) be a solution of (1) in E and let $(\tilde f, \tilde g)$ be a solution of (1) in $\tilde E$ with $E \subset \tilde E$; $(\tilde f, \tilde g)$ is an *extension* of (f, g) if f and g are the restrictions of $\tilde f$ and $\tilde g$ to $p_1(E) \cup p_2(E) \cup p_3(E)$.

Remark 1. If we study (f, g) as a solution of (1) in a triangle T_a or in a square Q_a, we can assume without loss of generality $a = 1$ and with reference to this *normalized* case we denote $I = (0, 1)$, $T = T_1$, $Q = Q_1$, $\xi = \xi_1$.

A solution (f, g) of (1) in E is *non-trivial* if $\Omega_g, \Omega_f \neq \emptyset$.

A solution (f, g) of (1) in E is *optimal* if $A_g \cap A_f = \emptyset$.

Remark 2. Let $E \subseteq \mathbb{R} \times \mathbb{R}$; (f, g) is a solution of (1) in E if and only if $\forall t \in p_3(E)$ and $\forall (x, t-x) \in E$

$$\xi_t(x) \neq g(t) \quad \text{implies} \quad \varphi_t(x) = f(t). \tag{2}$$

In particular (f, g) is a solution of (1) in T if and only if $G_t \cup F_t = (0, t)$, $\forall t \in (0, 1)$, where

$$G_t = \{x \in (0, t) : \xi_t(x) = g(t)\}, \quad F_t = \{x \in (0, t) : \varphi_t(x) = g(t)\}.$$

Remark 3. Let $(S, +) = (\mathbb{R}, +)$. Since (f, g) is a solution of (1) in $\mathbb{R}\times\mathbb{R}$ if and only if (h, k) defined by $h(x)=\lambda f(\alpha x)$, $k(x)=\mu g(\alpha x)$, $\lambda, \mu, \alpha\neq 0$, is, in the following section 4 we study, without loss of generality, the solutions of (1) in $\mathbb{R}\times\mathbb{R}$ such that the function g satisfies some regularity-hypothesis only in a *right* neighbourhood of the origin.

In order to present the aim of the present paper it is convenient to state before some considerations.

The previous results ([2]–[4]) which describe the non-trivial solutions of (1) are obtained starting from the following topological hypothesis on g:

$$p_i(\Omega_g) = p_i(\Omega_g^0) \quad i = 1, 2. \tag{3}$$

Thanks to these results we know that:

- ([3]) All solutions (f, g) of (1) in T, that satisfy condition (3), can be explicitely written and, by virtue of this, we can determine an extension $(\tilde{f}, \tilde{g})$ to $\mathbb{R} \times \mathbb{R}$ with $\tilde{g}$ still satisfying condition (3).
- ([4]) We can write the class of solutions of (1) in $\mathbb{R} \times \mathbb{R}$ satisfying condition (3) in T but not in Q and we know that their form depends on the group S.

As shown by an example in [2], the description of the solutions of equation (1) without any additional condition seems hopeless. Thus in the present paper we substitute condition (3) with some other algebraic and topological conditions.

In section 3 the problem of extending a solution of (1) given in T is treated and particular theorems, when g satisfies the condition

$$\exists \tau \in (0, 1) : \quad g(x + \tau) = g(x) + g(\tau), \; x \in (0, 1 - \tau) \tag{4}$$

are stated. To this purpose it is convenient to remark that if (f, g) is a solution of (1) in T then we know the values $f(x) + f(y)$ and $g(x) + g(y)$ also when $(x, y) \in Q \backslash \mathrm{T}$.

In section 4, we characterize classes of real and optimal solutions of (1) in $\mathbb{R} \times \mathbb{R}$, when g satisfies the following condition

$$L_t = \{t > 0 : T_t \subset A_g\} \neq \emptyset \quad \text{and } H_\alpha(\mathbb{R}^+) \supset U^+(0) \tag{5}$$
$$\text{where } \alpha = \operatorname{Sup} L_t.$$

3. On the extension of a solution (f,g)

In this section (f,g) denotes a non-trivial solution of (1) in T.

Lemma 1. *Let $t \in [1,2)$.*

i) There exists an extension $(\tilde{f},\tilde{g})$ of (f,g) to $T \cup (Q \cap r_t)$ if and only if there exists at least a couple of constant c_t and k_t such that, for every $x \in (t-1,1)$, $\xi_t(x) \neq c_t$ implies $\varphi_t(x) = k_t$; in this case we may assume $\tilde{g}(t) = c_t$ and $\tilde{f}(t) = k_t$.

ii) If ξ_t (or φ_t) is constant on $(t-1,1)$, then there exist infinitely many solutions in $T \cup (Q \cap r_t)$; otherwise there exist at most two solutions.

Proof. Since $(\tilde{f},\tilde{g})$ is a solution of (1) in $Q \cap r_t$ if and only if $\xi_t(x) \neq \tilde{g}(t)$ implies $\varphi_t(x) = \tilde{f}(t) \quad \forall x \in (t-1,1)$, i) follows easily.

ii). If $\xi_t(\varphi_t)$ is a constant function on $(t-1,t)$, all solutions are obtained by setting $\tilde{g}(t) = \xi_t$ and $\tilde{f}(t)$ arbitrarily chosen (or vice-versa). If there exists a solution in $Q \cap r_t$ when ξ_t and φ_t are not constant then, as a consequence of (i), there are two sets F and G with $F \cup G = (t-1,1)$ such that φ_t and ξ_t are constant functions on F and G respectively; as it is easy to see, this solution is not unique if and only if $F \cap G = \emptyset$; in this last case, φ_t and ξ_t are constant functions on G and F respectively. $\square$

Lemma 2. *Suppose $\xi(x) = c, \quad x \in I$.*

i) Let $(\tilde{f},\tilde{g})$ be an extension of (f,g) to T_2 with $\tilde{g}$ such that

$$\tilde{g}(t) = g(t-1) + c, \quad t \in (1,2) \quad \textit{and} \quad \tilde{g}(1) = c;$$

then

$$\tilde{f}(t) = f(t-1) + \varphi(t-1), \quad t \in p_i(\Omega_{\tilde{g}}) \cap (1,2),\ i = 1,2,3.$$

ii) Let $n \in \mathbb{Z}$ and let $(\tilde{f},\tilde{g})$ be defined by

$$\begin{cases} \tilde{f}(t) & = f(t-n) + n\varphi(t-n); \quad t \in (n,n+1) \\ \tilde{g}(t) & = g(t-n) + nc \end{cases}$$
and $\tilde{f}(n)$ arbitrary, $\tilde{g}(n) = nc$;

then $(\tilde{f},\tilde{g})$ is an extension of (f,g) to $\mathbb{R} \times \mathbb{R}$.

Proof. i). Since $T_2 = Q \cup (T_2 \backslash Q)$ and $p_i(\Omega_{\tilde{g}} \cap Q) \cap (1,2) = \emptyset$, $i = 1,2$, first we prove that i) holds with $t \in p_3(\Omega_{\tilde{g}} \cap Q) \cap (1,2)$. Let $x, y \in I$ with

$x+y=t$, $t\in p_3(\Omega_{\tilde g}\cap Q)\cap(1,2)$, then $\tilde g(t)=\tilde g(x+y)=g(x+y-1)+c\neq g(x)+g(y)$; from $\xi_{|I}=c$ it follows

$$\begin{cases} g(x+y-1)+g(1-x)\neq g(y)\\ g(x+y-1)+g(1-y)\neq g(x)\\ g(1-x)+g(1-y)\neq g(2-x-y)\end{cases}$$

therefore

$$\begin{cases} f(x+y-1)+f(1-x)=f(y)\\ f(x+y-1)+f(1-y)=f(x)\\ f(1-x)+f(1-y)=f(2-x-y).\end{cases}$$

Considering the above equations, we deduce $\varphi(x)=\varphi(y)=\varphi(x+y-1)$; the previous considerations and our hypothesis imply that

$$\begin{aligned}\tilde f(t)=\tilde f(x+y)&=f(x)+f(y)=\varphi(x)-f(1-x)+\varphi(y)-f(1-y)\\ &=2\varphi(x+y-1)-f(2-x-y)=\varphi(x+y-1)+f(x+y-1)\\ &=f(t-1)+\varphi(t-1).\end{aligned}$$

Now, we prove i) with $t\in p_i(\Omega_{\tilde g}\cap(T_2\backslash Q))\cap(1,2)$. Considering that if $i=3$ then $\exists x\in(1,2),\ y\in I$ with $t=x+y$ and $\tilde g(t)\neq\tilde g(x)+g(y)$ and if $i=1,2$ then $\exists y\in I$ with $t+y=x$ and $\tilde g(x)\neq\tilde g(t)+g(y)$, it is sufficient to prove that $\tilde g(t)=\tilde g(x+y)\neq\tilde g(x)+g(y)$ with $y\in I,\ x,t\in(1,2)$, implies

$$\tilde f(t)=f(t-1)+\varphi(t-1)\quad\text{and}\quad\tilde f(x)=f(x-1)+\varphi(x-1).$$

Since $\tilde g(x+y)=g(x+y-1)+c\neq g(x-1)+c+g(y)$ and $\xi(t)=c$, then

$$\begin{cases} g(1-y)\neq g(x-1)+g(2-x-y)\\ g(2-x)\neq g(2-x-y)+g(y)\\ g(x+y-1)+g(1-y)\neq\tilde g(x)\end{cases}$$

hence

$$\begin{cases} f(1-y)=f(x-1)+f(2-x-y)\\ f(2-x)=f(2-x-y)+f(y)\\ f(x+y-1)+f(1-y)=\tilde f(x).\end{cases}$$

By these equations it follows $\varphi(y)=\varphi(x-1)$ and $\tilde f(x)=f(x-1)+\varphi(x+y-1)$ and by virtue of $\tilde f(x+y)=\tilde f(x)+f(y)$, $f(x+y-1)=f(x-1)+f(y)$

and the previous considerations we have

$$\begin{aligned}\tilde{f}(t) &= \tilde{f}(x) + f(y) = f(x-1) + \varphi(x+y-1) + f(y)\\ &= f(x+y-1) + \varphi(x+y-1) = f(t-1) + \varphi(t-1),\\ \varphi(x+y-1) &= f(x+y-1) + f(2-x-y)\\ &= f(x-1) + f(y) + f(2-x-y) = \varphi(y) = \varphi(x-1)\end{aligned}$$

and so $\tilde{f}(x) = f(x-1) + \varphi(x-1)$.

ii). By the definition of g, it is easy to see that

$$\begin{gathered}(x,y) \in \Omega_{\tilde{g}} \iff (x-n, y-m) \in \Omega_{\tilde{g}} \quad n, m \in \mathbb{Z} \text{ and}\\ \{(x,y) : x \text{ or } y \text{ or } x+y \text{ belongs to } \mathbb{Z}\} \subseteq A_{\tilde{g}}\end{gathered}$$

hence, it is sufficient to show that $(\tilde{f}, \tilde{g})$ is a solution in $Q \backslash \overline{T}$.

So, let $x, y \in I,\ x+y = t \in (1,2)$ with $\tilde{g}(x+y) \neq g(x) + g(y)$: as shown in i), this implies

$$\begin{cases} f(x+y-1) + f(1-x) = f(y)\\ f(x+y-1) + f(1-y) = f(x)\\ f(1-x) + f(1-y) = f(2-x-y)\end{cases}$$

hence it follows $\varphi(x) = \varphi(y) = \varphi(x+y-1)$ and then

$$\begin{aligned}\tilde{f}(x+y) = f(x+y-1) + \varphi(x+y-1) = f(x+y-1) + \varphi(y) =\\ = f(x) - f(1-y) + \varphi(y) = f(x) + f(y).\end{aligned}$$

Thus the lemma is proved. $\square$

Theorem 1. *Suppose $\xi(x) = c$, $x \in I$ and let $\tilde{g}$ be defined by*

$$\tilde{g}(t) = g(t-n) + nc, \quad n \leq t < n+1; \quad \tilde{g}(0) = 0.$$

The pair $(\tilde{f}, \tilde{g})$ is an extension of (f, g) to $\mathbb{R} \times \mathbb{R}$ if and only if

$$\tilde{f}(t) = f(t-n) + n\varphi(t-n), \quad t \in p_i(\Omega_{\tilde{g}}), \quad i = 1,2,3,\ n < t < n+1,\ n \in \mathbb{Z}.$$

Proof. ii) of Lemma 2 shows that the condition is sufficient. Now we prove that it is also necessary.

By the definition of g and by $\xi(x) = c$ it follows easily that

$$t \in p_i(\Omega_{\tilde{g}}) \iff (t+n) \in p_i(\Omega_{\tilde{g}}) \quad \forall n \in \mathbb{Z}, \quad i = 1,2,3.$$

Furthermore, we have $\tilde{g}(x)+\tilde{g}(n-x)=nc \quad \forall n\in\mathbb{Z}$: indeed it is trivial if $x\in\mathbb{Z}$ and if $p<x<p+1$ with $p\in Z$ we have

$$\begin{aligned}\tilde{g}(x)+\tilde{g}(n-x)&=g(x-p)+pc+g(p+1-x)+(n-p-1)c\\&=g(x-p)+g(1-(x-p))+(n+1)c.\end{aligned}$$

Hence we can prove that $t\in p_i(\Omega_{\tilde{g}})$ for some $i=1,2,3$ if and only if $(n-t)\in p_k(\Omega_{\tilde{g}})$, $\forall n\in\mathbb{Z}$, for some $k=1,2,3$. To this purpose it is sufficient to observe that if $\tilde{g}(x+y)\neq\tilde{g}(x)+\tilde{g}(y)$ then $nc-\tilde{g}(n-(x+y))\neq nc-\tilde{g}(n-x)+\tilde{g}(y)$, hence $\tilde{g}(n-(x+y))+\tilde{g}(y)\neq\tilde{g}(n-x)$; this implies the previous property by putting $t=x+y$ or $t=x$.

To verify the necessary condition at first we show that the assertion is true

$$\forall t\in p_i(\Omega_{\tilde{g}})\cap(n,n+1),\ n\geq 0,\ i=1,2,3.$$

We proceed by induction. For $n=0$ it is trivial, for $n=1$ it is a consequence of ii) of Lemma 2. Assume it is true for $n-1$, $n\geq 2$, and let $t\in p_i(\Omega_{\tilde{g}})\cap(n,n+1)$. The pair $(\tilde{f},\tilde{g})$ is an extension of $(\tilde{f},\tilde{g})_{|T_n}$ and $\xi_n(x)=nc \quad \forall x\in(0,n)$; hence, using i) of Lemma 2 as in Remark 1 concerning the "normalized case", we can write

$$\tilde{f}(t)=f(t-n)+\varphi_n(t-n),\quad t\in p_i(\Omega_{\tilde{g}})\cap(n,2n),\quad i=1,2,3;$$

that is $\tilde{f}(t)=f(t-n)+f(t-n)+f(2n-t)$; since $2n-t\in(n-1,n)$, it follows

$$\begin{aligned}\tilde{f}(t)&=f(t-n)+f(t-n)+f(1+n-t)+(n-1)\varphi(1+n-t)\\&=f(t-n)+\varphi(t-n)+(n-1)\varphi(t-n)=f(t-n)+n\varphi(t-n).\end{aligned}$$

This proves the assertion $\forall t\in p_i(\Omega_{\tilde{g}})\cap(0,+\infty)$.

The next step is to show that

$$(x,y)\in\Omega_{\tilde{g}}\implies\varphi(x-n)=\varphi(y-m)=\varphi(x+y-r)$$

where $n,m,r\in\mathbb{Z}\ :\ n<x<n+1,\ m<y<m+1,\ r<x+y<r+1$. In fact by the property of $\tilde{g}$ we get

$$\begin{cases}\tilde{g}(x+y-n-m-1)+g(n+1-x)\neq g(y-m)\\ \tilde{g}(x+y-n-m-1)+g(m+1-y)\neq g(x-n)\end{cases}$$

then

$$\begin{cases}\tilde{f}(x+y-n-m-1)+f(n+1-x)=f(y-m)\\ \tilde{f}(x+y-n-m-1)+f(m+1-y)=f(x-n)\end{cases}$$

and this implies $\varphi(x-n)=\varphi(y-m)$; moreover, we can deduce that if $r=n+m$

$$\begin{cases} g(x+y-n-m) \neq g(x-n)+g(y-m) \\ -g[1-(x+y-n-m)] \neq g(x-n)-g[1-(y-m)], \end{cases}$$

that is

$$\begin{cases} f(x+y-n-m) = f(x-n)+f(y-m) \\ f[1-(x+y-n-m)] = -f(x-n)+f[1-(y-m)], \end{cases}$$

hence $\varphi(x+y-r)=\varphi(y-m)$. If $r=n+m+1$

$$\begin{cases} g(x+y-n-m-1)+g(1+m-y) \neq g(x-n) \\ -g[1-(x+y-n-m-1)]+g(1+m-y) \neq -g(1-(x-n)], \end{cases}$$

and so, by proceeding in the same way, we obtain $\varphi(x+y-r)=\varphi(x-n)$.

At last we show the assertion for

$$t \in p_i(\Omega_{\tilde{g}}) \cap (-\infty, 0), \quad i=1,2,3; \quad -n<t<-n+1, \quad n \geq 1.$$

If $i=1,2$ then $\exists y \in (n,n+1) : (t,y),(t+n,y-n) \in \Omega_{\tilde{g}}$; therefore $\tilde{f}(t+y)=\tilde{f}(t)+\tilde{f}(y)$ and $\tilde{f}(t+y)=f(t+n)+f(y-n)$ hold; this implies $\tilde{f}(t)-f(t+n)=f(y-n)-\tilde{f}(y)$ and, having $y>0$, $\tilde{f}(y)=f(y-n)+n\varphi(y-n)=f(y-n)+n\varphi(x+n)$ and so $\tilde{f}(t)=f(t+n)-n\varphi(t+n)$.

If $i=3$ then $\exists(x,y) \in \Omega_{\tilde{g}}$, $x+y=t$ with $r<x<r+1$, $s<y<s+1$ and $-n=r+s,\ r+s+1$ for suitable $r,s \in \mathbb{Z}$; since $(x-r,y-s)$ is in $\Omega_{\tilde{g}}$ as well, by virtue of the previous property we can write

$$\begin{aligned} \tilde{f}(t) &= \tilde{f}(x)+\tilde{f}(y) = f(x-r)+r\varphi(x-r)+f(y-s)+s\varphi(y-s) \\ &= \tilde{f}(x+y-r-s)+(r+s)\varphi(t+n) \end{aligned}$$

hence we obtain $\tilde{f}(t)=f(t+n)-n\varphi(t+n)$ in the case $-n=r+s$; in the case $-n=r+s+1$ we deduce

$$\tilde{f}(t)=\tilde{f}(t+1+n)-(n+1)\varphi(t+n)=f(t+n)-n\varphi(t+n).$$

The necessary condition is so proved. $\square$

Lemma 3. *Let* $0 < x < t < 1 - \tau$ *and assume condition* (4) *holds; then*

i) $\xi_{t+\tau}(x) = \xi_{t+\tau}(x+\tau) = \xi_t(x) + g(\tau)$, $\varphi_{t+\tau}(x) = \varphi_t(x) + \Delta_\tau f(t-x)$ *and* $\varphi_{t+\tau}(x+\tau) = \varphi_t(x) + \Delta_\tau f(x)$.

ii) $x \notin G_t \Rightarrow \Delta_\tau f(x) = \Delta_\tau f(t) = \Delta_\tau f(t-x)$

Proof. i) is easily verified.
ii). By i) and Remark 2, we have $x \in G_t \Leftrightarrow x \in G_{t+\tau} \Leftrightarrow x+\tau \in G_{t+\tau}$ therefore $x \notin G_t$ implies that $f(t) = \varphi_t(x)$, $f(t+\tau) = \varphi_{t+\tau}(x) = \varphi_{t+\tau}(x+\tau)$ hold and then $f(t+\tau) - f(t) = \varphi_{t+\tau}(x) - \varphi_t(x) = \varphi_{t+\tau}(x+\tau) - \varphi_t(x)$, so by i) and by the definition of Δ_τ we deduce ii). □

Lemma 4. *Let* $n\tau < 1$, $n \geq 2$ *and assume condition* (4) *holds; then*

$$f(t+q\tau) = f(t) + q\Delta_\tau f(t), \quad t \in p_3(\Omega_g), \quad t+q\tau \in (0,1), \quad q \in \mathbb{Z}.$$

Proof. In this case the functions $\Delta_\tau g$ and $\Delta_\tau f$ are obviously defined in $(0,\tau] \subset (0, 1-\tau)$ and $\Delta_\tau g(x) = g(\tau)$.

The assertion is trivial if $q = 0$ or $q = 1$.

Let $t \in p_3(\Omega_g)$, $t+q\tau \in (0,1)$, $q \geq 2$; by (4) we can suppose $t \in p_3(\Omega_g \cap Q_\tau)$. So, $\exists (x,y) \in (0,\tau)$, $x+y = t \;:\; x, y \notin G_t;\; x, x+\tau, x+2\tau \notin G_{t+2\tau}$ and then

$$\begin{aligned} f(t+2\tau) &= \varphi_{t+2\tau}(x+\tau) = f(x+\tau) + f(t+\tau-x) \\ &= f(x) + \Delta_\tau f(x) + f(t-x) + \Delta_\tau f(t-x) \\ &= \varphi_t(x) + \Delta_\tau f(x) + \Delta_\tau f(t-x) \end{aligned}$$

and, by Lemma 3, $\quad f(t+2\tau) = f(t) + 2\Delta_\tau f(t)$.

As a consequence of i) of Lemma 3, we can write

$$\begin{aligned} f(t+2\tau) &= \varphi_{t+2\tau}(x+2\tau) = \varphi_{t+\tau}(x+\tau) + \Delta_\tau f(x+\tau) \\ &= \varphi_t(x) + \Delta_\tau f(x) + \Delta_\tau[f(x) + \Delta_\tau f(x)] \\ &= f(t) + 2\Delta_\tau f(x) + \Delta_\tau[f(x+\tau) - f(x)] \\ &= f(t) + 2\Delta_\tau f(t) + \Delta_\tau[f(t+\tau) - f(t)] \end{aligned}$$

and therefore $\Delta_\tau[f(t+\tau) - f(t)] = 0$.

Now we proceed by induction; let the assertion be true for $q \geq 2$; since for $t + (q+1)\tau \in (0,1)$ we have

$$\begin{aligned} f(t+(q+1)\tau) &= f(t+q\tau) + \Delta_\tau f(t+q\tau) \\ &= f(t) + q\Delta_\tau f(t) + \Delta_\tau[f(t) + q\Delta_\tau f(t)] \\ &= f(t) + (q+1)\Delta_\tau f(t) + q\Delta_\tau[f(t+\tau) - f(t)] \\ &= f(t) + (q+1)\Delta_\tau f(t) \end{aligned}$$

and the lemma is so proved. $\square$

Theorem 2. *Let $\Delta_\tau f = k$, k constant, and assume that condition* (4) *holds with $\tau \le \frac{1}{2}$. Then $(\tilde{f}, \tilde{g})$ defined by*

$$\begin{cases} \tilde{f}(t) = f(t - n\tau) + nk \\ \tilde{g}(t) = g(t - n\tau) + ng(\tau) \end{cases} \qquad n\tau < t \le (n+1)\tau,\ n \in \mathbb{Z} \tag{6}$$

is an extension of (f, g) to $\mathbb{R} \times \mathbb{R}$.

Proof. It is easy to see that $\tilde{f}_{|I} = f$ and $\tilde{g}_{|I} = g$. Now, if $(\tilde{f}, \tilde{g})$ is a solution in $\overline{T}_{2\tau}$ then it is also a solution in $\mathbb{R} \times \mathbb{R}$; indeed $\tilde{g}(x+y) \ne \tilde{g}(x) + \tilde{g}(y)$, with $n\tau < x \le (n+1)\tau$, $m\tau < y \le (m+1)\tau$, $(n+m)\tau < x+y \le (n+m+2)\tau$, $n, m \in \mathbb{Z}$ implies that $\tilde{g}(x+y-(n+m)\tau)) \ne g(x-n\tau)+g(y-m\tau)$, therefore $\tilde{f}(x+y-(n+m)\tau)) = f(x-n\tau)+f(y-m\tau)$ and so, by adding to both sides of this equality $nk+mk$, we get $\tilde{f}(x+y) = \tilde{f}(x) + \tilde{f}(y)$.

Thanks to the previous consideration, the theorem is proved when $\tau < \frac{1}{2}$, because in this case $(\tilde{f}, \tilde{g})_{|}\overline{T}_{2\tau} = (f, g)$ holds; when $\tau = \frac{1}{2}$ it is sufficient to verify that $(\tilde{f}, \tilde{g})$ is a solution on $\{x, y > 0 : x + y = 1\}$. To this purpose, by using (2), we can show that

$$\xi(x) \ne \tilde{g}(1) = g(1-\tau) + g(\tau) \quad \text{implies} \quad \varphi(x) = \tilde{f}(1) = f(1-\tau) + k.$$

Let $x \in (0,1) : \xi(x) \ne \tilde{g}(1)$, it is obviously $x \ne k$ and we can suppose $x \in (0, \tau)$, having $\xi(x+\tau) = \xi(x)$;

$$g(x) + g(1-x) \ne g(\tau) + g(1-\tau) \quad \text{implies} \quad g(x) + g(1-\tau-x) \ne g(1-\tau)$$

and the following equality

$$f(x) + f(1-\tau-x) = f(1-\tau), \quad f(x) + f(1-x) - \Delta_\tau f(1-x-\tau) = f(1-\tau)$$

that is $\varphi(x) = f(1-\tau) + k = \tilde{f}(1)$. $\square$

Remark 4. Let (f, g) be the solution in T with g and f defined by

$$g(x) = \begin{cases} 0, & x \in [0, 3/5) \\ 1, & x \in [3/5, 4/5)\ . \\ 0, & x \in [4/5, 1) \end{cases} \qquad f(x) = \begin{cases} 1, & x \in [0, 3/5) \\ 2, & x \in [3/5, 4/5) \\ 3, & x \in [4/5, 1). \end{cases}$$

In this case (4) holds with $\tau = 4/5$, and $\Delta_\tau f = 2$ is a constant function; since $(3/5, 3/5) \in \Omega_{\tilde{g}} \cap \Omega_{\tilde{f}}$ this example shows that Theorem 2 is not true if $\tau > 1/2$.

Theorem 3. *Let (f,g) be such that*

$$(x,y)\in\Omega_g\cap(Q_\tau\backslash T_\tau) \implies x,y\in G_\tau \text{ or } x,y,x+y\notin G_\tau$$

and assume (4) *holds with $\tau<1/2$. Then $(\tilde f,\tilde g)$ defined by*

$$\text{(6')}\qquad \begin{cases}\tilde f(t)=f(t-n\tau)+n\Delta_\tau f(t-n\tau)\\ \tilde g(t)=g(t-n\tau)+ng(\tau)\end{cases}$$

$$n\tau<t\le(n+1)\tau,\ n\in\mathbb{Z},\ t\ge 1$$

is an extension of (f,g) to $\mathbb{R}\times\mathbb{R}$.

Proof. Let $(x,y)\in\Omega\tilde g$; (4) and (6') imply that $x,y\neq n\tau\quad\forall n\in\mathbb{Z}$. Let $n\tau<x<(n+1)\tau,\quad m\tau<y<(m+1)\tau,\quad(m+m)\tau<x+y<(m+m+2)\tau$; since $(x-n\tau,y-m\tau)\in\Omega_g$ then $f(x+y-(n+m)\tau)=f(x-n\tau)+f(y-m\tau)$.

If $x+y\le(m+m+1)\tau$, it follows

$$\begin{aligned}\tilde f(x+y)&=f(x+y-(n+m)\tau)+(n+m)\Delta_\tau f(x+y-(n+m)\tau)\\&=f(x-n\tau)+f(y-m\tau)+(n+m)\Delta_\tau f(x+y-(n+m)\tau)\end{aligned}$$

and, by virtue of ii) of Lemma 3

$$\Delta_\tau f(x+y-(n+m)\tau)=\Delta_\tau f(x-n\tau)=\Delta_\tau f(y-m\tau)$$

and so $(x,y)\in A_{\tilde f}$.

If $x+y>(m+m+1)\tau$, it follows

$$\begin{aligned}\tilde f(x+y)&=f(x+y-(n+m+1)\tau)\\&\qquad+(n+m+1)\Delta_\tau f(x+y-(n+m+1)\tau)\\&=f(x+y-(n+m)\tau)+(n+m)\Delta_\tau f(x+y-(n+m+1)\tau)\\&=f(x-n\tau)+f(y-m\tau)+(n+m)\Delta_\tau f(x+y-(n+m+1)\tau)\end{aligned}$$

and the assertion proving that

$$\Delta_\tau f(x+y-(n+m+1)\tau)=\Delta_\tau f(x-n\tau)=\Delta_\tau f(y-m\tau).$$

To this purpose, let $u=x-n\tau,\ v=y-m\tau$; the previous considerations imply that $(u,v)\in\Omega_g\cap(Q_\tau\backslash T_\tau),\quad g(u+v-\tau)+g(\tau)\neq g(u)+g(v)$ and by the hypothesis that $u,v\in G_\tau$ or $u,v,u+v-\tau\notin G_\tau$.

In the first case, when $\xi_\tau(u)=\xi_\tau(v)=g(\tau)$, we obtain

$$g(u+v-\tau)+g(\tau-u)\neq g(v),\qquad g(u+v-\tau)+g(\tau-v)\neq g(u)$$

and as consequence of ii) of Lemma 3 $\Delta_\tau f(u+v-\tau) = \Delta_\tau f(u) = \Delta_\tau f(v)$. In the second case, by virtue of ii) of Lemma 3, we have

$$\Delta_\tau f(u), \Delta_\tau f(v), \Delta_\tau f(u+v-\tau) = \Delta_\tau f(\tau).$$

The theorem is so proved. □

4. Optimal solutions of (1) in $\mathbb{R}\times\mathbb{R}$

In this paragraph we suppose $(X,+) = (S,+) = (\mathbb{R},+)$. In the following $[t]$ denotes the integral part of t and $[t]_* = -(1+[-t])$.

Lemma 5. *Let $\alpha,\beta,c,d \in \mathbb{R}\ :\ 0<\alpha<\beta\ ;\ c,d\neq 0$. The pair (f,g) defined by*

$$\left\{\begin{aligned} f(x) &= \left(\left[\tfrac{x}{\alpha}\right] + \left[\tfrac{x}{\beta}\right] + 1\right) d\\ g(x) &= \left(\left[\tfrac{x}{\alpha}\right] - \left[\tfrac{x}{\beta}\right]\right) c\end{aligned}\right. \tag{7}$$

or by

$$\left\{\begin{aligned} f(x) &= \left(\left[\tfrac{x}{\alpha}\right] + \left[\tfrac{x}{\beta}\right]_* + 1\right) d\\ g(x) &= \left(\left[\tfrac{x}{\alpha}\right] - \left[\tfrac{x}{\beta}\right]_*\right) c\end{aligned}\right. \tag{8}$$

is a non-trivial optimal solution of (1) in $\mathbb{R}\times\mathbb{R}$.

Proof. Let (f,g) be of the form (7) (the proof is the same for the form (8)) and let

$$x \in [i\alpha,(i+1)\alpha)\cap[r\beta,(r+1)\beta),\quad y\in[j\alpha,(j+1)\alpha)\cap[s\beta,(s+1)\beta),$$
$$i,j,r,s\in\mathbb{Z}.$$

So $x+y \in [m\alpha,(m+1)\alpha)\cap[n\beta,(n+1)\beta)$ with $m=i+j$ or $m=i+j+1$ and $n=r+s$ or $n=r+s+1$.

Now, since

$$g(x)+g(y) = (i+j-r-s)c,\quad f(x)+f(y) = (i+j+2+r+s)d$$
$$g(x+y) = (m-n)c,\quad f(x+y) = (m+n+1)d$$

we obtain

$$g(x+y)\neq g(x)+g(y) \iff \begin{cases} m=i+j\\ n=r+s+1\end{cases} \text{or} \quad \begin{cases} m=i+j+1\\ n=r+s\end{cases}$$

therefore, $g(x+y)\neq g(x)+g(y)$ if and only if $f(x+y)=f(x)+f(y)$. □

Definition. Two solutions (f_1, g_1), (f_2, g_2) of equation (1) are equivalent if $f_2 - f_1$ and $g_2 - g_1$ are additive functions.

Lemma 6. *Any non-trivial optimal solution of* (1) *in* $\mathbb{R}\times\mathbb{R}$, *for which condition* (5) *with* $H_\alpha \neq \mathbb{R}^+$ *holds, is equivalent to a solution of the form* (7) *or* (8).

Proof. By the condition (5) and the results in [3], we may assume $g(x) = 0$ and so

$$\xi_\alpha(x) = g(x) + g(\alpha - x) = 0 \quad , \quad x \in (0, \alpha).$$

Let $g(\alpha) = c$ and let α' be defined by $\alpha' = \mathrm{Sup}\{t > 0 \ : \ (0,t) \subset H_\alpha\}$. From the definition of α and the assumption $H_\alpha \neq \mathbb{R}^+$, it follows

$$g(x) = ic, \ x \in [i\alpha, (i+1)\alpha) \cap (0, \alpha + \alpha') \text{ with } c \neq 0 \text{ and } \alpha' < +\infty.$$

Let $\beta = \alpha + \alpha'$ and $N_0 \in N$, $N_0 \geq 1 \ : \ N_0\alpha \leq \beta < (N_0 + 1)\alpha$; the form of g implies that $\Omega_g(= A_f)$ contains all points $(x, y) \in T_\beta$ such that

$$x \in [i\alpha, (i+1)\alpha), \ y \in [j\alpha, (j+1)\alpha), \ x + y \in [(i+j+1)\alpha, (i+j+2)\alpha)$$

with $i, j \geq 0 \ : \ i + j = 0, 1, \dots, N_0 - 1$. Therefore ([3]) we may assume $f(x) = (i+1)d$, $x \in [i\alpha, (i+1)\alpha) \cap (0, \beta)$ with $d \neq 0$ and $f(\alpha) = \varphi_\alpha(x) = 2d$, $x \in (0, \alpha)$. Now, let $x, y \in (0, \beta)$, $x \in [i\alpha, (i+1)\alpha)$, $y \in [j\alpha, (j+1)\alpha)$; since we have

$$g(x) + g(y) = (i + j)c, \qquad f(x) + f(y) = (i + j + 2)d,$$

if $t \in [\beta, 2\beta) \cap [m\alpha, (m+1)\alpha)$, $t = x + y$, $(x, y) \in Q_\beta \backslash T_\beta$, we have that

$$(\xi_t, \varphi_t) \in \{(mc, (m+2)d) \ ; \ ((m-1)c, (m+1)d)\}.$$

So, by virtue of ii) of Lemma 1, necessarily it is

$$(9) \quad \begin{cases} f(t) = (m+2)d \\ g(t) = (m-1)c \end{cases} \qquad \text{or} \qquad (9') \quad \begin{cases} f(t) = (m+1)d \\ g(t) = mc \end{cases}$$

Now we prove that if $t \in (\beta, 2\beta)$ then (9) holds.
The definition of β implies that $\forall \epsilon > 0 \ \exists \gamma \in [\beta, \beta + \epsilon) \cap [N_0\alpha, (N_0 + 1)\alpha))$ such that $g(\gamma) \neq g(\gamma - \alpha) + g(\alpha)$, therefore $f(\gamma) = f(\gamma - \alpha) + f(\alpha)$ and so

$$\begin{cases} f(\gamma) = (N_0 + 2)d \\ g(\gamma) = (N_0 - 1)c. \end{cases}$$

Suppose, ab absurdo, that $\exists t_0 \in (\beta, 2\beta) \cap [m\alpha, (m+1)\alpha)$ such that

$$\begin{cases} f(t_0) = (m+1)d \\ g(t_0) = mc. \end{cases}$$

It follows $t_0 - \gamma \in (l\alpha, (l+1)\alpha) \cap (0, \beta)$ with $l = m - N_0 - 1$ or $l = m - N_0$ and therefore we have

$$f(t_0 - \gamma) + f(\gamma) = (m+2)d \quad \text{or} \quad f(t_0 - \gamma) + f(\gamma) = (m+3)d$$

and

$$g(t_0 - \gamma) + g(\gamma) = (m-2)c \quad \text{or} \quad g(t_0 - \gamma) + g(\gamma) = (m-1)c;$$

so $(t_0 - \gamma, \gamma) \in \Omega_g \cap \Omega_f$ contrary to the hypothesis.
Now, since

$$\begin{cases} f(\beta) = (N_0+2)d \\ g(\beta) = (N_0-1)c \end{cases} \quad \text{or} \quad \begin{cases} f(\beta) = (N_0+1)d \\ g(\beta) = N_0 c, \end{cases}$$

by virtue of the previous considerations we see that $(f,g)_{|T_{2\beta}}$ is of the form (7) or (8) respectively. The next step is to show that if $(f,g)_{|T_{2\beta}}$ is of the form (7), then $(f,g)_{|T_{n\beta}}$, $n \geq 2$, is of the same form and so is (f,g) on the whole $\mathbb{R} \times \mathbb{R}$. By induction, let $(f,g)_{|T_{n\beta}}$ be of the form (7) and consider $t \in [m\alpha, (m+1)\alpha) \cap [n\beta, (n+1)\beta)$; since if $(x,y) \in \{T_{(n+1)\beta} \cap Q_{n\beta}\} \backslash T_{n\beta}$ is such that

$$x \in [i\alpha, (i+1)\alpha) \cap [r\beta, (r+1)\beta), \ y \in [j\alpha, (j+1)\alpha) \cap [s\beta, (s+1)\beta)$$

with $0 \leq r, \ s < n$, we obtain $m = i + j$ or $m = i + j + 1$, $n = r + s$ or $n = r + s + 1$ and

$$g(x) + g(y) = (i + j - r - s)c; \qquad f(x) + f(y) = (i + j + 2 + r + s)d;$$

we can remark, as above in $Q_\beta \backslash T_\beta$, that

$$(\xi_t, \varphi_t) \in \Big\{ \big((m-n)c, (m+n+2)d\big) \ ; \ \big((m-n+1)c, (m+n+1)d\big);$$
$$\big((m-1-n)c, (m+n+1)d\big) \ ; \ \big((m-n)c, (m+n)d\big) \Big\}$$

and therefore, by virtue of Lemma 1, $f(t) = (m+n+1)d$ and $g(t) = (m-n)c$ necessarily holds.

So $(f,g)_{|T_{(n+1)\beta}}$ is of the form (7) and therefore $(f,g)_{|\mathbb{R}^+ \times \mathbb{R}^+}$ as well.

At last, if (f,g) is of the form (7) in $\mathbb{R}^+ \times \mathbb{R}^+$ we prove that (f,g) is of the same form in $\mathbb{R} \times \mathbb{R}$.

Let $y < 0,\ y \in [j\alpha, (j+1)\alpha) \cap [s\beta, (s+1)\beta),\ j, s < 0$. Since $\forall x > -y$ with $x \in [i\alpha, (i+1)\alpha) \cap [r\beta, (r+1)\beta),\ x+y \in [m\alpha, (m+1)\alpha) \cap [n\beta, (n+1)\beta)$ we have

$$g(x+y) - g(x) = (m-n-i+r)c \quad \text{and} \quad f(x+y) - f(x) = (m+n-i-r)d$$

with $j = m-i$ or $j = m-i-1,\ s = n-r$ or $s = n-r-1$; we can remark that

$$\big(g(x+y) - g(x), f(x+y) - f(x)\big) \in \big\{\big((j-s)c, (j+s)d\big); \\ \big((j-s-1)c, (j+s+1)d\big); \big((j+1-s)c, (j+s+1)d\big); \\ \big((j-s)c, (j+s+2)d\big)\big\}$$

and consequently $f(y) = (j+s+1)d$ and $g(y) = (j-s)c$ necessarily holds. So (f, g) is of the form (7) in $(\mathbb{R}\backslash\{0\}) \times (\mathbb{R}\backslash\{0\})$. With similar considerations, as consequence of the following equalities

$$\begin{array}{lll} g(x) + g(-x) = 0, & f(x) + f(-x) = 0, & x \in (0, \alpha) \\ g(\alpha) + g(-\alpha) = c, & f(\alpha) + f(-\alpha) = d & \\ g(\beta) + g(-\beta) = -c, & f(\beta) + f(-\beta) = d, & \text{if} \quad N_0\alpha < \beta < (N_0+1)\alpha \\ g(\beta) + g(-\beta) = 0, & f(\beta) + f(-\beta) = 2d, & \text{if} \quad \beta = N_0\alpha \end{array}$$

we obtain $g(0) = 0$ and $f(0) = d$; so (7) holds in $\mathbb{R} \times \mathbb{R}$. With a similar prove we show that if $(f, g)_{|T_{2\beta}}$ is of the form (8) then (f, g) is of the same form in $\mathbb{R} \times \mathbb{R}$. □

Theorem 4. *The pair (f, g) is a non-trivial optimal solution of* (1) *in $\mathbb{R} \times \mathbb{R}$, satisfyng* (5) *with $H_\alpha \neq \mathbb{R}^+$, if and only if it is equivalent to a solution of the form* (7) *or* (8).

Proof. Lemma 6 proves the necessary condition.
Let (f, g) defined by (7) or (8); by Lemma 5, (f, g) is a solution of (1), furthermore it is easy to see that (5) holds with $H_\alpha \supset (0, \beta - \alpha)$ and we have $\mathbb{R}^+ \neq H_\alpha$ because $H_\alpha \cap (\beta - \alpha, 2\beta - \alpha) = \emptyset$; hence the sufficient condition is proved. □

Remark 5. Let $\beta = \frac{p}{q}\alpha,\ p, q \in \mathbb{N},\ p > q$ *and* $\tau = q\beta = p\alpha$; then

if (f, g) is defined by (7) it follows $g(x+\tau) = g(x) + g(\tau), \qquad x \in \mathbb{R}$;
if (f, g) is defined by (8) it follows $f(x+\tau) = f(x) + f(\tau), \qquad x \in \mathbb{R}$.

Indeed, in the first case $x \in [i\alpha, (i+1)\alpha) \cap [r\beta, (r+1)\beta)$ implies

$$x + \tau \in [(i+p)\alpha, (i+p+1)\alpha) \cap [(r+q)\beta, (r+q+1)\beta)$$

and therefore

$$g(x+\tau) = (i+p-r-q)c = g(x) + g(\tau);$$

in the second case $x \in [i\alpha, (i+1)\alpha) \cap (r\beta, (r+1)\beta]$ implies

$$x + \tau \in [(i+p)\alpha, (i+p+1)\alpha) \cap ((r+q)\beta, (r+q+1)\beta]$$

and therefore

$$f(x+\tau) = (i+p+1+r+q)d = (i+r+1)d + (p+q)d = f(x) + f(\tau).$$

Remark 6. By Remark 3 and Theorem 4, we can characterize in a similar way the solutions of (1) in $\mathbb{R} \times \mathbb{R}$ having the form

$$\text{(7')} \qquad \begin{cases} f(x) = \left(\left[\dfrac{x}{\alpha}\right]_* + \left[\dfrac{x}{\beta}\right]_* + 1 \right) d \\ g(x) = \left(\left[\dfrac{x}{\alpha}\right]_* - \left[\dfrac{x}{\beta}\right]_* \right) c \end{cases}$$

or

$$\text{(8')} \qquad \begin{cases} f(x) = \left(\left[\dfrac{x}{\alpha}\right]_* + \left[\dfrac{x}{\beta}\right] + 1 \right) d \\ g(x) = \left(\left[\dfrac{x}{\alpha}\right]_* - \left[\dfrac{x}{\beta}\right] \right) c \end{cases}$$

Remark 7. For the solutions of (1), characterized by Theorem 4, condition (3) is not satisfied because

$$\{x = p\alpha : p\alpha \neq q\beta,\ p, q \in \mathbb{N}\} \subset (p_i(\Omega_g) \backslash p_i(\Omega_g)^0) \quad i = 1, 2.$$

Remark 8. If $\beta = N_0\alpha = 1,\ N_0 > 1$, then (f, g) defined by (7) or (7') satisfies the condition (3) in T but not in Q([4]).

Lemma 7. *Any non-trivial optimal solution of* (1) *in* $\mathbb{R} \times \mathbb{R}$, *for which condition* (5) *with* $H_\alpha = \mathbb{R}^+$ *holds, is equivalent to a solution having one of the following forms* (10) *or* (11)*:*

$$\text{(10)} \qquad \begin{cases} f(x) = \left(\left[\dfrac{x}{\alpha}\right] + 1 \right) d \\ g(x) = \left[\dfrac{x}{\alpha}\right] c \end{cases}$$

$$(11) \qquad \begin{cases} f(x) = \left[\dfrac{x}{\alpha}\right] d \\ g(x) = \left(\left[\dfrac{x}{\alpha}\right] + 1\right) c \end{cases} \textit{for } x < 0,$$

$$\begin{cases} f(x) = \left(\left[\dfrac{x}{\alpha}\right] + 1\right) d \\ g(x) = \left[\dfrac{x}{\alpha}\right] c \end{cases} \textit{for } x > 0$$

$$\begin{cases} f(0) = d \\ g(0) = 0 \end{cases} \textit{or} \quad \begin{cases} f(0) = 0 \\ g(0) = c, \end{cases}$$

$\alpha, c, d \in \mathbb{R}$ *with* $\alpha > 0$ *and* $c, d \neq 0$.

Proof. Proceeding as in Lemma 6, it is easy to see that $H_\alpha = \mathbb{R}^+$ implies $\alpha' = +\infty = \beta$, $T_\beta = Q_\beta = \mathbb{R}^+ \times \mathbb{R}^+$ and since in $\mathbb{R}^+ \times \mathbb{R}^+$ (10) and (11) coincide, the assertion is proved in $\mathbb{R}^+ \times \mathbb{R}^+$.

To prove the assertion in $\mathbb{R} \times \mathbb{R}$, let $y < 0$, $y \in [j\alpha, (j+1)\alpha)$; since $\forall x > -y$ with $x \in [i\alpha, (i+1)\alpha)$, $x + y \in [m\alpha, (m+1)\alpha)$ we have

$$g(x+y) - g(x) = (m-i)c \qquad \text{and} \qquad f(x+y) - f(x) = (m-i)d$$

with $j = m - i$ or $j = m - i - 1$, we can remark that

$$(g(x+y) - g(x), f(x+y) - f(x)) \in \{(jc, jd); ((j+1)c, (j+1)d)\}$$

and consequently

$$(10\text{"}) \quad \begin{cases} f(y) = (j+1)d \\ g(y) = jc \end{cases} \qquad \text{or} \quad (11\text{"}) \quad \begin{cases} f(y) = jd \\ g(y) = (j+1)c. \end{cases}$$

Let A and B be the subsets of $\mathbb{R}^-$ in which (10") and (11") are verified respectively; it is $A \cap B = \emptyset$ and $A \cup B = \mathbb{R}^-$ manifestly.
Since we can remark that if $x, y < 0$ then

$$(g(x) + g(y), f(x) + f(y)) = ((i+j)c, (i+j+2)d) \text{ in } A \times A,$$
$$(g(x) + g(y), f(x) + f(y)) = ((i+j+2)c, (i+j)d) \text{ in } B \times B,$$
$$(g(x) + g(y), f(x) + f(y)) = ((i+j+1)c, (i+j+1)d) \text{ in } A \times B;$$

we deduce $B = \emptyset$ or $A = \emptyset$, that is (10) or (11) holds in $(\mathbb{R}\backslash\{0\}) \times (\mathbb{R}\backslash\{0\})$.
If (10) is true in $(\mathbb{R}\backslash\{0\}) \times (\mathbb{R}\backslash\{0\})$, the following equalities

$$g(x) + g(-x) = -c, \; f(x) + f(-x) = d, \qquad x \in (i\alpha, (i+1)\alpha)$$

and

$$g(i\alpha) + g(-i\alpha) = 0, \quad f(i\alpha) + f(-i\alpha) = 2d$$

imply that $g(0) = 0$ and $f(0) = d$ necessarily holds, so (10) is true in $\mathbb{R} \times \mathbb{R}$.

If (11) is true in $(\mathbb{R}\backslash\{0\}) \times (\mathbb{R}\backslash\{0\})$, the following equalities

$$g(x) + g(-x) = 0, \; f(x) + f(-x) = 0, \qquad x \in (i\alpha, (i+1)\alpha);$$

and

$$g(i\alpha) + g(-i\alpha) = c, \quad f(ia) + f(-i\alpha) = d$$

imply that $g(0) = 0$ and $f(0) = d$ or $g(0) = c$ and $f(0) = 0$ necessarily holds, so the assertion is true. $\square$

Theorem 5. *The pair* (f, g) *is a non-trivial optimal solution of* (1) *in* $\mathbb{R} \times \mathbb{R}$*, satisfyng* (5) *with* $H_\alpha = \mathbb{R}^+$*, if and only if it is equivalent to a solution of the form* (10) *or* (11)*.*

Proof. Lemma 7 proves the necessary condition. It is easy to verify the sufficient condition.

Remark 9. Let (10') and (11') be obtained by exchanging in (10) and (11) the symbols [ ] and $[\ ]_*$.

As in Remark 6, we can characterize in a similar way the solutions of (1) in $\mathbb{R} \times \mathbb{R}$ defined by (10') or (11').

Remark 10. Theorem 5 gives, besides the solutions defined by (10) (and already determined in [3] under the condition (3)), also the solutions defined by (11) for which

$$\{x = i\alpha : i \in \mathbb{Z}\backslash\{0\}\} \subset (p_i(\Omega_g)\backslash p_i(\Omega_g^0)), \quad i = 1, 2.$$

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DANIELA RUSCONI
DIPARTIMENTO DI MATEMATICA
UNIVERSITA DI MILANO
VIA SALDINI 50
I–20133 MILANO, ITALY

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