Univalent functions convex in one direction

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1. Introductory Remarks

A domain D in the complex plane is convex in one direction (the direction of the imaginary axis) if for every pair of points (u, v_1) and (u, v_2) in D the set of points $(u, tv_1+(1-t)v_2)$, $0<t<1$, is also in D. A function $f(z)$, regular and univalent in the unit disc E, is called convex in one direction if $f(E)$ is convex in one direction. This class of functions was introduced by FEJÉR [1] and studied more extensively by ROBERTSON [5], and [6].

In 1936 Robertson proved the following theorem [6].

Theorem A. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular in E and suppose $f(z)$ satisfies one of the following conditions:*

(i) *$f(z)$ is regular on $|z|=1$ and $f(E)$ is convex in one direction.*

(ii) *if $E_r=\{z:\ |z|<r,\ 0<r\leqq 1\}$, then there exists a positive $\delta=\delta(f)$ so that for all r in $1-\delta<r\leqq 1$, $f(E_r)$ is convex in one direction.*

Then there exist real numbers μ and ν, $0\leqq\mu\leqq\pi$ and $0\leqq\nu\leqq\pi$,so that

$$\operatorname{Re}\{-ie^{i\mu}(1-2\cos\nu\, e^{-i\mu}z+e^{-2i\mu}z^2)f'(z)\}\geqq 0. \tag{1.1}$$

The numbers μ and ν have a geometrical interpretation which will be discussed below.

Conditions (i) and (ii) in the above theorem are restrictive in the sense that one can give examples of functions $f(z)$ which do not satisfy either condition such that $f(E)$ is convex in one direction.

Recently W. HENGARTNER and G. SCHOBER obtained a condition similar to (1.1) by dispensing with the assumptions of regularity on the boundary and convexity in one direction of the level curves and assuming the normalization that $z=\pm 1$ correspond, in some sense, to the left and right extremes of $f(E)$. If $f(z)$ is regular in E and $|z_0|=1$, then $f(z_0)$ is a right (left) extreme of $f(E)$ if there is a sequence of points $\{z_n\}$ in E with the property that $\lim_{n\to\infty} z_n=z_0$ and $\lim_{n\to\infty}\operatorname{Re}\{f(z_n)\}=\sup_E \operatorname{Re}\{f(z)\}$ $(\inf_E \operatorname{Re}\{f(z)\})$. The following is a summary of results due to Hengartner and Schober which appear in [3].

Theorem B. *Let $f(z)$ be regular and non-constant in E. Suppose $f(z)$ maps E univalently onto a domain D convex in one direction. Then:*

(i) *if the prime ends $f(1)$ and $f(-1)$ are the right and left extremes of D, then*

$$\operatorname{Re}\{(1-z^2)f'(z)\} \geqq 0, \quad z \in E. \tag{1.2}$$

(ii) *if $f(1)$ is both the right and left extreme of D, and there is at least one vertical ray in the complement of D which meets ∂D from above, then*

$$\operatorname{Im}\{(1-z)^2 f'(z)\} \geqq 0, \quad z \in E. \tag{1.3}$$

(iii) *if $f(-1)$ is both the right and left extreme of D, and there is at least one vertical ray in the complement of D which meets ∂D from below, then*

$$\operatorname{Im}\{(1-z)^2 f'(z)\} \leqq 0, \quad z \in E. \tag{1.4}$$

Conversely, if $f(z)$ satisfies (1.2), (1.3) *or* (1.4), *then $f(z)$ maps E univalently onto a domain convex is one direction with the corresponding normalization.*

Conditions (i), (ii), and (iii) are also restrictive in that functions $f(z)$ can be found which do not satisfy any of the three normalization requirements and which map E onto a domain convex in one direction.

The main result of this paper is to show that the above restrictions can be removed and that Robertson's condition (1.1) is the proper one provided certain interpretations are given to the parameters μ and ν. In Section 2 we prove that any function which is convex in one direction satisfies (1.1) for appropriate μ and ν and, conversely, any function satisfying (1.1) must map E univalently onto a domain convex in one direction. The parameters μ and ν are used to decompose the class of all functions convex in one direction into subclasses. Some mapping properties of functions in these subclasses are studied in Section 3.

2. Principal Result

Theorem 1. *Let $f(z)$ be a non-constant function regular in E. The function $f(z)$ maps E univalently onto a domain D convex in the direction of the imaginary axis if and only if there are numbers μ and ν, $0 \leqq \mu < 2\pi$ and $0 \leqq \nu \leqq \pi$, such that*

$$\operatorname{Re}\{-ie^{i\mu}(1-2\cos\nu\, e^{-i\mu}z+e^{-2i\mu}z^2)f'(z)\} \geqq 0. \quad z \in E. \tag{2.1}$$

Furthermore, $f(e^{i(\mu-\nu)})$ and $f(e^{i(\mu+\nu)})$ are the right and left extremes, respectively, of D.

PROOF. Suppose $f(z)$ maps E univalently onto a domain D convex in one direction. Since the theorem is obviously true for $f(z)=z$, we can assume $f(z)$ is not the identity function and D is not the entire plane. Let $f(z_1)$ and $f(z_2)$ correspond to right and left extremes of D. Choose μ and ν so that $z_1=e^{i(\mu-\nu)}$ and $z_2=e^{i(\mu+\nu)}$, $0 \leqq \mu < 2\pi$ and $0 \leqq \nu \leqq \pi$. If $z_1 \neq z_2$, then $\nu \neq 0$ or π and it is possible to construct a function $w(z)$ mapping E onto E so that $w(z_1)=1$ and $w(z_2)=-1$. A general expression for such a function is given by

$$w(z) = ie^{i(\gamma-\mu)}(z-t)/(1-\bar{t}z) \tag{2.2}$$

where $e^{i\gamma}=(1-i\alpha\beta)/(1+i\alpha\beta)$, $t=e^{i\mu}(\alpha-i\beta)/(1-i\alpha\beta)$, $\alpha=\cos v/(1+\sin v)$, and β is an arbitrary real number satisfying $-1<\beta<1$. A direct calculation shows that $w(z_1)=1$, $w(z_2)=-1$. Furthermore, by proper choice of β it is possible to make any point on the upper half of $|w|=1$ the preimage of any point $e^{i\varphi}$ where $\mu-v<\varphi<\mu+v$, that is, any function which maps E onto E and satisfies the requirement that z_1 is the preimage of $w=1$ and z_2 is the preimage of $w=-1$ can be expressed in the form given in (2.2) for a suitable choice of β. Now let $g(z)$ be defined by $f(z)=g(w(z))$. Since $f(z)$ and $w(z)$ are both univalent in E, it follows that $g(z)$ is univalent in E and maps E onto D. Furthermore $g(z)$ satisfies condition (i) of Theorem B, hence Re $\{(1-z^2)g'(z)\}\geqq 0$. A direct calculation shows that

$$(1-w(z)^2)g'(w(z)) = [(1-w(z)^2)/w'(z)]f'(z) =$$

$$=(-ie^{i\mu}/\sin v)(1-2ze^{-i\mu}\cos v+z^2e^{-2i\mu})f'(z)$$

which, since $z_1\neq z_2$ implies $0<v<\pi$, yields (2.1). Now assume $z_1=z_2=e^{i\mu}$. Since D is convex in one direction and is not the entire plane, there must be at least one vertical ray in the complement of D issuing from the boundary of D. If this ray meets ∂D from above then applying condition (ii) of Theorem B to the function $g(z)=f(e^{-i\mu}z)$ yields (2.1). If the ray meets ∂D from below, an application of part (iii) of Theorem B to $g(z)=f(e^{i(\pi-\mu)}z)$ gives (2.1).

Conversely, if $f(z)$ satisfies (2.1), then the function $g(z)$, as defined above, will satisfy the hypothesis of Theorem B which implies that $g(z)$ maps E univalently onto a domain convex in one direction and consequently $f(z)$ will do the same.

This shows that Robertson's condition (1.1) is the correct for any univalent function convex is one direction and that hypothesis (i) and (ii) are superfluous

3. Mapping Properties

Let Γ represent the class of normalized univalent functions which map E onto domains convex in one direction, that is, $f(z)$ is in Γ if and only if $f(0)=0$, $f'(0)=1$ and $f(z)$ satisfies (2.1) for some choice of μ and v. With the exception of a possible translation and magnification, every simply connected domain convex in one direction will be the image of some function in Γ. The parameters μ and v offer a convenient and interesting decomposition of Γ into subclasses. Let $\Gamma(v,\mu)$ be the class of all functions in Γ which satisfy (2.1) for a given pair μ and v and let $\Gamma(v)=$ $=\bigcup_\mu \Gamma(v,\mu)$. The condition $f'(0)=1$ implies that Re$\{-ie^{+i\mu}\}\geqq 0$, hence μ must satisfy $0\leqq\mu\leqq\pi$ when $f(z)$ is normalized.

If $\mathscr{P}$ is the class of regular functions $p(z)$ with positive real part in E and normalized by $p(0)=1$, then (2.1) implies that $f(z)$ is in $\Gamma(v,\mu)$ if and only if

$$zf'(z) = h_v(e^{-i\mu}z)[\cos\mu+i\sin\mu\, p(z)] \tag{3.1}$$

when $h_v(z)=z(1-2\cos v\, z+z^2)^{-1}$, $p(z)\in\mathscr{P}$, $0\leqq v\leqq\pi$ and $0\leqq\mu\leqq\pi$. It is interesting to note that the class $\Gamma(v,0)$ consists only of the function $f(z)=\int_0^z (h_v(t)/t)\,dt$ and $\Gamma(v,\pi)$ has as its only member $-f(-z)$.

The number v can be considered a measure of the distance between the two points on the unit circle which in some sense maximize Re $\{f(z_1)-f(z_2)\}$. If $f(z)$ is in $\Gamma(v)$ and z_1 and z_2 are the points which correspond to the right and left extremes of $f(E)$, then $|\arg z_1-\arg z_2|=2\min\{v, \frac{\pi}{2}-v\}$ where the arguments of z_1 and z_2 are chosen so as to make the absolute value of their difference as small as possible. The choice of v is not necessarily unique since the function $f(z)=i/2\log[(i+z)/(i-z)]$ is in $\Gamma(v)$ for every $v\neq 0$.

The remainder of this paper is concerned with determining some of the mapping properties of $\Gamma(v)$. It is convenient to begin with some preliminary lemmas concerning the representation (3.1) since it is fundamental in studying the mapping properties.

Lemma 1. Let $h_v(z)=z/(1-2\cos v\, z+z^2)$, $0\leqq v\leqq\pi$, *then*

$$(3.2)\quad [r/(1+2r|\cos v|+r^2)]\leqq|h_v(z)|\leqq\begin{cases}[r/(1-2r|\cos v|+r^2)], & r<(1-\sin v)/|\cos v|\\ [r/\sin v(1-r^2)], & (1-\sin v)/|\cos v|\leqq r<1,\end{cases}$$

with the interpretation that the first inequality on the right is used for all r when $v=0$ or π and the second is used for all r when $v=\pi/2$, and

$$(3.3)\qquad \operatorname{Re}\{zh_v'(z)/h_v(z)\}\geqq(1-r^2)/(1+2|\cos v|r+r^2),\quad |z|\leqq r.$$

Proof. A brief calculation shows that

$$(3.4)\qquad g(\theta)=|h_v(re^{i\theta})|^2=r^2[1+4r^2\cos^2 v-2r^2+r^4+4r^2K(\cos\theta)]^{-1},$$

where $K(x)=x^2-ax$, $a=(1+r^2)\cos v/r$ and $|x|\leqq 1$. Thus the maximum value o. $K(x)$ is $1+|a|$ and the minimum value is $-a^2/4$ when $|a|\leqq 2$ and $1-|a|$ when $|a|\geqq 2$. Substitution of these values into (3.4) immediately yields (3.2). Since $zh_v'(z)/h_v(z)=$ $=(1-z^2)/(1-2bz+z^2)$, $b=\cos v$, (3.3) is equivalent to the following inequalities:

$$2r(1+\cos\theta)[b(1-2br+r^2)+2r(1-\cos\theta)]\geqq 0,\quad b\geqq 0,$$

$$2r(\cos\theta-1)[b(1+2br+r^2)-2r(\cos\theta+1)]\geqq 0,\quad b\leqq 0,$$

both of which are clearly valid for all r and θ.

Lemma 2. *If $p(z)$ is in $\mathcal{P}$ and $0\leqq\mu\leqq\pi$, then*

$$(3.5)\qquad (1-r)/(1+r)\leqq|\cos\mu+i\sin\mu p(z)|\leqq(1+r)/(1-r),\quad |z|\leqq r$$

with equality when $\mu=\pi/2$ and $p(z)=(1+z)/(1-z)$.

Lemma 3. *If $p(z)$ is in $\mathcal{P}$ and β is real, then*

$$(3.6)\qquad |zp'(z)/(p(z)+i\beta)|\leqq 2r/(1-r^2),\quad |z|\leqq r,$$

with equality when $\beta=0$ and $p(z)=(1+z)/(1-z)$.

Lemma 2 follows directly from the fact that $p(z)$ is subordinate to $(1+z)/(1-z)$ and lemma 3 can be found in [4].

Theorem 2. *If $f(z)$ is in $\Gamma(\nu)$, then*

$$(3.7)\qquad (1-r)/(1+r)(1+2r|\cos\nu|+r^2) \leqq |f'(z)|,$$

$$(3.8)\qquad |f'(z)| \leqq \begin{cases}(1+r)/(1-r)(1+2r|\cos\nu|+r^2), & r<(1-\sin\nu)/|\cos\nu| \\ 1/\sin\nu(1-r)^2, & (1-\sin\nu)/|\cos\nu| \leqq r<1\end{cases}$$

where the inequalities in (3.8) *are treated in the same manner as those in* (3.2) *when* $\nu=0,\pi$, *or* $\pi/2$. *These results are the best possible.*

Proof. (3.7) and (3.8) follow directly from (3.1), (3.2) and (3.5). Equality occurs in (3.7) and the first inequality in (3.8) for $f(z)$ defined by $f'(z)=(1+\varepsilon z)/(1+2i\cos\nu z-z^2)(1-\varepsilon z)$ where $\varepsilon=-i$ if $\cos\nu\geqq 0$ and $\varepsilon=i$ if $\cos\nu<0$. Equality occurs in the second inequality in (3.8) for the same function $f(z)$ at $z=iz_0$ where $z_0=(1/2)[(1+r^2)\cos\nu+i\sqrt{4r^2-(1+r^2)^2\cos^2\nu}]$ and $\varepsilon=-i\bar{z}_0/r$.

Theorem 3. *If $f(z)$ is in $\Gamma(\nu)$, then for $0<\nu<\pi$*

$$(3.9)\qquad \log[(1+r)/(1+2|\cos\nu|r+r^2)^{1/2}]/(1-|\cos\nu|) \leqq |f(z)|,$$

$$\begin{matrix}(3.10)\\(3.11)\end{matrix}\quad |f(z)| \leqq \begin{cases}\log[(1-2|\cos\nu|r+r^2)^{1/2}/(1-r)]/(1-|\cos\nu|), & r<(1-\sin\nu)/|\cos\nu| \\ r/\sin\nu(1-r), & (1-\sin\nu)/|\cos\nu| \leqq r<1\end{cases}$$

and for $\nu=0$ or π

$$(3.12)\qquad r/(1+r)^2 \leqq |f(z)| \leqq r/(1-r)^2, \quad |z|\leqq r.$$

Proof. The upper bounds follow directly from (3.8) by integrating the maximum value of $|f'(z)|$ and the lower bound follows from integrating the minimum value of $|f'(z)|$ along the inverse of the segment joining the origin to the point on $f(|z|=r)$ closest to the origin. [See for example [3]]. Inequalities (3.9), (3.10), and (3.12) are sharp for $f(z)$ defined by $f'(z)=(1+\varepsilon z)/(1+2i\cos\nu z-z^2)(1-\varepsilon z)$, $\varepsilon=\pm i$. (3.11) is not sharp, but the order is correct since $h_\nu(z)$ is in $\Gamma(\nu)$ and $|h_\nu(z)|=O(1-r)^{-1}$.
Letting $r\to 1$ in (3.9) yields the following.

Corollary. If $f(z)$ is in $\Gamma(\nu)$ then $f(E)$ contains the disk

$$|w| < \{\log[2/(1+|\cos\nu|)]\}/2(1-|\cos\nu|) = \varrho_\nu, \quad (1/2)\log 2 \geqq \varrho_\nu \geqq 1/4.$$

Theorem 4. *The radius of convexity of $\Gamma(\nu)$ is*

$$(3.13)\qquad r_\nu = \frac{1}{2}\{1+[5+4|\cos\nu|]^{1/2}-[2+4|\cos\nu|+2(5+4|\cos\nu|)^{1/2}]^{1/2}\}.$$

Proof. Differentiating (3.1) logarithmically we obtain

$$\mathrm{Re}\{1+zf''(z)/f'(z)\} = \mathrm{Re}\{e^{-i\mu}zh_\nu'(e^{-i\mu}z)/h_\nu(e^{-i\mu}z)\}+\mathrm{Re}\{zP'(z)/[p(z)-i\cot\mu]\} \geqq$$
$$\geqq [1-2r-2(1+2|\cos\nu|)r^2-2r^3+r^4]/(1-r^2)(1+2r|\cos\nu|+r^2),$$

where we have used (3.3) and (3.6) and have assumed $\mu\neq 0$. This last assumption is no restriction since $\Gamma(\nu,0)$ contains only one function, $f_\nu(z)=\int_0^z[h_\nu(t)/t]\,dt$, and this function is convex in the entire unit disc. Evidently, $f(z)$ maps $|z|<r$ onto a convex domain whenever $1-2r-2(1+2|\cos v|)r^2-2r^3+r^4>0$. A brief calculation shows r_v as given in (3.13) is the smallest positive root of the above polynomial. If $f(z)$ is defined by

$$f'(z) = (1-iz)/(1+2\cos v\; iz-z^2)(1+iz), \quad \cos v \geqq 0,$$

then $f(z)$ is in $\Gamma(v)$ and

$$\left(1+\frac{zf''(z)}{f'(z)}\right)_{z=-ir} = \frac{1-2r-2r^2-4r^2\cos v-2r^3+r^4}{(1-r^2)(1+2r\cos v+r^2)} = 0$$

when $r=r_v$, thus $f(z)$ is not convex in any larger disk. If $\cos v<0$ then $g(z)= -f(-z)$ is in $\Gamma(v')$ where $v'=\pi/2-v$ and, since $\cos v=\cos v'$, $g(z)$, and hence $f(z)$, is not convex in any disk with radius greater than r_v.

Theorem 5. *If* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ *is in* $\Gamma(v)$,

$$L_r = \int_0^{2\pi} |zf'(z)|\,d\theta \leqq \begin{cases} \sqrt{2}\,\pi/\sin v\,(1-r) & v \neq 0, \pi \\ \pi/(1-r)^2, & v = 0 \text{ or } \pi \end{cases}$$

and

$$|a_n| \leqq \begin{cases} 1 & v = \pi/2 \\ 2\sqrt{2}/\sin v, & 0 < |v-\pi/2| < \pi/2 \\ n, & v = 0 \text{ or } \pi. \end{cases}$$

PROOF. Let $zf'(z)=h_v(e^{-i\mu}z)[\cos\mu+i\sin\mu\, p(z)]$, where

$$h_v(z)=z+\sum_{k=2}^{\infty}(\sin kv/\sin v)z^k \text{ and } p(z)=1+\sum_{k=1}^{\infty}p_k z^k.$$

$$\int_0^{2\pi} |h_v(z)|^2\,d\theta = 2\pi\sum_{k=1}^{\infty}(\sin^2 kv/\sin^2 v)r^{2\mu} = \frac{2\pi^2(1+r^2)}{(1-r^2)(1-2r^2\cos 2v+r^4)}$$

and, using the fact that $|p_k|\leqq 2$,

$$\int_0^{2\pi} |\cos\mu+i\sin\mu\, p(z)|^2\,d\theta = 2\pi\left(1+\sum_{k=1}^{\infty}|\sin\mu|^2|p_k|^2r^{2k}\right) \leqq 2\pi\left(1+4\sum_{k=1}^{\infty}r^{2k}\right) = \frac{2\pi(1+3r^2)}{1-r^2}.$$

Thus

$$L_r = \int_0^{2\pi} |zf'(z)|\,d\theta = \int_0^{2\pi} |h_v(e^{-i\mu})|\cdot|\cos\mu+i\sin\mu\, p(z)|\,d\theta \leqq$$

$$\leqq \left\{\int_0^{2\pi} |h_v(e^{-i\mu}z)|^2\,d\theta\cdot\int_0^{2\pi}|\cos\mu+i\sin\mu\, p(z)|^2\,d\theta\right\}^{1/2} \leqq$$

$$\leqq \frac{2\pi}{1-r}\left\{\frac{r^2(1+3r^2)}{(1+r)^2(1-2r^2\cos 2v+r^4)}\right\}^{1/2}.$$

Now $r^2(1+3r^2)(1+r)^{-2}(1-2r^2\cos 2v+r^4)^{-1}<(\sqrt{2}\sin v)^{-1}$ $[v\neq 0$ or $\pi]$ and $<1/2(1-r)^2$ $[v=0$ or $\pi]$ yields the desired result. The inequality $|a_n|\leqq 1$ for $v=\pi/2$ follows from a theorem in [3] and $|a_n|\leqq n$ for $v=0$ or π is obvious since $f(z)$ is close -to-convex. Both inequalities are sharp. We have not been able to determine the sharp bounds on $|a_n|$ when $0<|v-\pi/2|<\pi/2$, however the bound on L_r shows that the coefficient of f are bounded, since

$$n|a_n| \leqq 1/(2\pi r^n)\int_0^{2\pi}|zf'(z)| \leqq \sqrt{2}/2\sin v\, r^n(1-r)$$

which implies

$$|a_n| \leqq (\sqrt{2}/2\sin v)\big(1+1/(n-1)\big)^n \leqq 2\sqrt{2}/\sin v$$

where $r=1-1/n$.

It is possible to obtain a smaller asymptotic bound for $|a_n|$ by applying a lemma due to GRONWALL [2] which states if $M_n(v)=\frac{1}{n}\sum_{k=1}^{n}|\sin kv|$, then $M(v)=\lim_{n\to\infty}M_n(v)$ exists and equals $2/\pi$ if v/π is irrational; $M(\pi k/l)=[\cot(\pi/2l)]/l<2/\pi$, if $k, l(k<l)$ are relatively prime positive integers. Indeed, it follows from (3.1) that

$$na_n = \frac{\sin nv}{\sin v}e^{-i(n-1)\mu}+i\sin\mu\sum_{k=1}^{n-1}\frac{\sin kv}{\sin v}e^{-ik\mu}p_{n-k},$$

so that

$$|na_n| \leqq \left|\frac{\sin nv}{\sin v}\right|+2\sin\mu\left[1+\sum_{k=2}^{n-1}\left|\frac{\sin kv}{\sin v}\right|\right], \tag{3.14}$$

where $0\leqq\mu\leqq\pi$, $0\leqq v\leqq\pi$. Hence

$$|a_n| \leqq 1+2\sin\mu\left(\frac{n-1}{2}\right) = 1+(n-1)\sin\mu$$

and if $\varrho\neq 0$, π we have using the lemma and (3.14) that

$$\overline{\lim_{n\to\infty}}|a_n| \leqq \frac{2\sin\mu}{\sin v}M(v) \leqq \frac{4}{\pi}\frac{\sin\mu}{\sin v} \leqq \frac{4}{\pi\sin v}.$$

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(Received August 3, 1973; in revised form July 10, 1974.)