A certain radius of convexity problem

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Introduction. Let E denote the unit disc and C denote the class of functions f regular in E normalised by $f(0)=f'(0)-1=0$ and which satisfies the condition that it maps E onto a domain which is convex. It is interesting to ask whether

$$f \text{ and } g\in C \text{ imply } \lambda f+(1-\lambda)g\in C \text{ for } 0<\lambda<1.$$

(Cf. Problem 6.11 in [2].). This problem could be answered in the negative. However the radius of convexity of the family consisting of $\lambda f+(1-\lambda)g$ for arbitrary $f, g\in C$ and each fixed $\lambda\in(0, 1)$ is still not known. A more general problem would be to ask for the radius of convexity of $\lambda f+(1-\lambda)g$ where f is convex and g is starlike, (with respect to origin). Since a complete solution is not known, it will be interesting to consider special cases in which the problem admits a complete solution. We can weaken the question either by specialising f or g or both. In this context the choice of $g=zf'$ is interesting, since the starlike function associated to an $f\in C$ is zf', in the sense that $f\in C$ iff zf' is starlike (with respect to origin). We specialise f also by requiring that $p(z)=[1+(zf''(z)/f'(z)]$ should satisfy

$$|(p(z)-1)(p(z)+1)^{-1}|<\alpha \quad \text{for} \quad z\in E \quad \text{and} \quad 0<\alpha\leqq 1.$$

We also choose $\lambda=2^{-1}$.

Both the choices $\lambda=2^{-1}$ and the speciality of f indicated above are motivated by the fact that the result obtained may be a natural refinement of the one obtained by LIVINGSTON in [3]. The technique we adopt is suggested by the work in [4]. Thus in this paper we find out the radius of convexity of the family $C(\alpha)$ which can be described as follows.

$f(z)\in C(\alpha)$ iff f is regular in E and satisfies $f(0)=0, f'(0)=1$ and $|p(z)-1)(p(z)+1)^{-1}|<\alpha$ where $p(z)=[1+(zf''/f')]$ and then

$$P(\alpha)=\{g(z)=2^{-1}[f(z)+zf'(z)]/f(z)\in C(\alpha)\}.$$

Theorem. *The radius of convexity γ^* of the family $C(\alpha)$ can be descirbed as follows.*

(i) *For* $0<\alpha\leqq[(\sqrt{5}+1)/4]$, $\gamma^*=[(p-1+\alpha)/2\alpha]^{1/2}$ *where* $p^2=(1-\alpha)(1+3\alpha)$.

(ii) *For* $[\sqrt{5}+1/4]\leqq\alpha\leqq 1$, $\gamma^*=(2\alpha)^{-1}$.

PROOF. Let $g(z)=2^{-1}[f(z)+zf'(z)]$, $f(z)\in C(\alpha)$ $p(z)=\left[1+\left(zf''(z)/f'(z)\right)\right]$, $p^*(z)=\left[1+\left(zg''(z)/g'(z)\right)\right]$, and $-w(z)=\left(p(z)-1\right)\alpha^{-1}\left(p(z)+1\right)^{-1}$, $r=|z|$. From $g\in P(\alpha)$ we have $|w(z)|<1$ for $z\in E$. $w(0)=0$ and so by Schwarz lemma $|w(z)|\leqq r$ $(z\in E)$. Simple calculation shows that

$$\text{(1)} \qquad \operatorname{Re} p^*(z) = -1+2\operatorname{Re} q(z)-\operatorname{Re} \alpha z w'(z) q(z)$$

where $q(z)=[1+\alpha w(z)]^{-1}$.

From $|w(z)|\leqq r$ we have

$$|q(z)-a| \leqq d \quad \text{where} \quad a=(1-\alpha^2r^2)^{-1}, \quad d=\alpha r a.$$

Now $\Phi(z)=z^{-1}w(z)$ satisfies $|\Phi(z)|\leqq 1$ $(z\in E)$ and so by a well known result [1],

$$|\Phi'(z)| \leqq (1-r^2)^{-1}\left(1-|\Phi(z)|^2\right).$$

So, $|zw'(z)-w(z)|\leqq\left(r^2-|w(z)|^2\right)(1-r^2)^{-1}$ and thus $\operatorname{Re} zw'(z)q(z)\leqq\operatorname{Re} w(z)q(z)+\left(r^2-|w(z)|^2\right)(1-r^2)^{-1}|q(z)|$. Using these result and (1) elementary calculations give

$$\text{(2)} \qquad \operatorname{Re} p^*(z) \geqq -2+3(a+u)-(1-\alpha^2r^2)(1-r^2)^{-1}[(a+u)^2+v^2]^{-1/2}(d^2-u^2-v^2)$$

where u and v real numbers depending on z, $q(z)=a+u+iv$, $u^2+v^2\leqq d^2$. If we denote the right side of (2) which is a function of u and v by $s(u, v)$ we find that

$$\partial s(u,v)|\partial v = \alpha^{-1}(1-r^2)^{-1}(1-\alpha^2r^2)vR^{-4}T(u,v)$$

where $R=|q(z)|$ and $T(u, v)>0$.

So the right side of (2) considered as a function of v with u fixed takes its minimum for $u^2+v^2\leqq d^2$ at $v=0$ and so,

$$\text{(3)} \qquad \operatorname{Re} p^*(z) \geqq -2+3(a+u)-(1-\alpha^2r^2)(1-r^2)^{-1}\alpha^{-1}(a+u)^{-1}(d^2-u^2).$$

Now we consider the right side of (3) as a function $F(u)$ of $u\geqq -d$, with r fixed and calculate the number $u_0=u_0(r)$ where F takes its absolute minimum. Since $|u|\leqq d$ in (3) and F takes its minimum for $|u|\leqq d$ at u_0, we have,

$$\text{(4)} \qquad \operatorname{Re} p^*(z) \geqq F(u_0) \quad \text{if} \quad u_0 \geqq -d.$$

If $u_0<-d$, F is a monotonic increasing function of u and so,

$$\text{(5)} \qquad \operatorname{Re} p^*(z) \geqq F(-d) \quad \text{if} \quad u_0 < -d.$$

Now g is convex for $|z|<\varrho$ if $\operatorname{Re} p^*(z)>0$ for $|z|<\varrho$. Thus we can get (i) and (ii) from (4) and (5) by elementary calculations which can be left to the reader.

We can see (ii) is sharp for each α by considering the function f defined by

$p(z)=(1-\alpha z)(1+\alpha z)^{-1}$. In the case of (i), consider the function f defined by $w(z)=z(z-t)(1-tz)^{-1}$ where t is determined by the condition $1+\alpha w(r_1)=[u_0(r_1)+a]^{-1}$,

$$r_1^2 = (2\alpha)^{-1}(p-1+\alpha).$$

The detailed calculation is again left to the reader.

In conclusion the author would like to thank Professor K.S. PADMANABHAN for helpful discussions. His thanks are also due to the *referee** for valuable suggestions towards improvement.

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(Received March 22, 1974; in revised form July 14, 1975.)

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