Modules that are finite sums of simple submodules

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A module M over a commutative ring R is said to be completely reducible [4, *p*. 167] if each submodule of M is a direct summand of M. Each submodule of a completely reducible module is completely reducible, and in a completely reducible module, the following five conditions are equivalent:

(1) M is Noetherian.

(2) M is Artinian.

(3) M has finite length.

(4) M is a finite sum of simple submodules, where a module N is *simple* if (0) and N are the only submodules of N.

(5) M is a finite direct sum of simple submodules.

A finite sum of simple modules is completely reducible [4, p, 168], and hence each submodule of a finite sum of simple modules is again a finite sum of simple modules. It is the purpose of this paper to consider the following question.

If each proper submodule of M is a finite sum of simple modules, is M a finite sum of simple modules?

We prove that he answer to the preceding question is negative, and in Theorem 1 we determine, to within isomorphism, those modules for which each the proper submodule is a finite sum of simple modules. First we need a description of the simple R-modules; if X is a nonempty subset of an R-module M, then $\mathrm{Ann}(X)$ denotes the annihilator of X — that is, $\mathrm{Ann}(X)=\{r\in R|rx=0$ for each x in $X\}$. The following result is well known [4, p. 133].

Result 1. *A nonzero commutative ring R has no ideals other than* (0) *and R if and only if either R is the zero ring on a cyclic group of prime order or R is a field.*

There is an analogue of Result 1 for noncommutative rings [3; Exercise 2. p. 101]:

A nonzero associative ring S has only two right ideals, (0) *and S, if and only if either S is the zero ring on a cyclic group of prime order or S is a division ring.*

Result 1 enables us to determine all nonzero simple modules over a ring R.

Proposition 1. *Let M be a nonzero module over a commutative ring R. In order that M be simple, it is necessary and sufficient that one of the following conditions* (1) *or* (2) *is satisfied:*

(1) *Scalar multiplication is trivial and the additive group of* M *is cyclic of prime order.*

(2) *Scalar multiplication is not trivial,* $R/\mathrm{Ann}(M)$ *is a field, and* M *is a one-dimensional vector space over* $R/\mathrm{Ann}(M)$ *under the induced scalar multiplication.*

PROOF. It is clear that M is simple if (1) or (2) is satisfied. We show that if M is simple and (1) fails, then condition (2) is satisfied. Since (1) fails, scalar multiplication is nontrivial, for the only nonzero simple abelian groups are those of prime order. Thus, let m be an element of M such that $\mathrm{Ann}(m) \neq R$. Since M is simple, $M = Rm$ and the mapping $r \to rm$ is an R-module homomorphism of R onto M with kernel $\mathrm{Ann}(M)$. Therefore $R/\mathrm{Ann}(M)$ and M are isomorphic R-modules as well as isomorphic $(R/\mathrm{Ann}(M))$-modules. It then follows from Result 1 that $R/\mathrm{Ann}(M)$ is a field and that M is a one-dimensional vector space over $R/\mathrm{Ann}(M)$.

We note that in seeking to determine conditions under which each proper submodule of an R-module M is a finite sum of simple modules, there is no loss of generality in assuming that R has an identity element and M is unitary, for if R^* is the ring obtained from R by canonically adjoining an identity of characteristic 0 (see [1] or [2; p. 5]), then M is uniquely a unitary R^*-module in such a way that the scalar multiplication between elements of R and elements of M is retained [2; Exercise 4, p. 9], and under this multiplication the R-submodules of M are the same as the R^*-submodules of M. Hence in our statement of Theorem 1 we assume that R has an identity element and M is unitary.

Theorem 1. *Assume that* R *is a commutative ring with identity and* M *is a unitary* R*-module. If each proper submodule of* M *is a finite sum of simple submodules of* M*, then either* (1) M *is a finite sum of simple submodules, or* (2) M *is cyclic,* $R/\mathrm{Ann}(M)$ *is a local ring with maximal ideal* $P/\mathrm{Ann}(M)$*, and* $P^2 \subseteqq M$*. Conversely, each proper submodule of* M *is a finite sum of simple submodules if* (1) *or* (2) *is satisfied.*

PROOF. We assume that M is nonzero and that each proper submodule of M is a finite sum of simple submodules of M. Thus if N is a nonzero proper submodule of M, then $N = S_1 + \dots + S_t$, where each S_i is nonzero and simple. Proposition 1 implies that $P_i = \mathrm{Ann}(S_i)$ is a maximal ideal of R for each i, and hence $\mathrm{Ann}(N) = \bigcap_1^t P_i$ is a finite intersection of maximal ideals of R. Let $\{P_\alpha\}$ be the family of maximal ideals of R, and for each α, let N_α be the submodule of M consisting of elements of M annihilated by P_α. As we have just observed, ΣN_α contains each proper submodule of M. Hence, *if* M *is not cyclic,* then $M = \Sigma N_\alpha$. We show that this implies that M is a finite sum of simple submodules; we consider separately the cases where exactly one, or more than one, of the submodules N_α is nonzero.

Case I. $M = N_\alpha$ for some α. Then to within isomorphism, M is a vector space over R/P_α. Since each proper subspace of M is finite-dimensional, M itself is finite-dimensional. Thus M is a finite sum of simple submodules.

Case II. $M \supset N_\alpha$ for each α. We choose α so that $N_\alpha \neq (0)$ and we show that $M = N_\alpha \oplus \sum_{\beta \neq \alpha} N_\beta$. Thus if $m \in N_\alpha \cap (\Sigma N_\beta)$, then for some finite set $\{\beta_i\}_1^n$ of elements $\beta \neq \alpha$, $m \in N_\alpha \cap (N_{\beta_1} + \dots + N_{\beta_n})$; hence $P_\alpha m = 0 = (P_{\beta_1} \cap \dots \cap P_{\beta_n})m$, and since

$P_\alpha+(P_{\beta_1}\cap\ldots\cap P_{\beta_n})=R$, $Rm=0$ and $m=0$. It follows that N_α and $\sum_{\beta\neq\alpha} N_\beta$ are proper submodules of M, and hence are finite sums of simple submodules. Consequently, M is a finite sum of simple submodules.

We have proved that condition (1) is satisfied if M is not cyclic. If M is cyclic, then as in the proof of Proposition 1, M is isomorphic, as an R-module or as an $(R/\mathrm{Ann}\,(M))$ module, to $R/(\mathrm{Ann}\,(M))$. Hence to complete the proof of the first half of Theorem 1, we consider a ring S with identity such that each proper ideal of S is a finite sum of minimal ideals of S, and we seek to prove that either S is a finite sum of minimal ideals or S is local with nonzero maximal ideal P such that $P^2=(0)$. If S has more than one maximal ideal, then S can be written as the sum of two proper ideals, and hence S is a finite sum of minimal ideals. Assume that S is quasi-local with maximal ideal P. Since P has finite length as an S-module, S also has finite length, so S is Noetherian; that is, S is a local ring. We have already observed that the annihilator of xS, for each nonzero element x of P, is a finite intersection of maximal ideals of S, and hence is P; consequently $P^2=(0)$. Thus either $P=(0)$, S is a field, and S is a finite sum of minimal ideals, or we have the second of the desired conditions satisfied.

We turn to a proof of the converse, which is quite easy. As previously remarked, (1) implies that each submodule of M is a finite sum of simple submodules, and if (2) is satisfied, then we need only observe that each proper ideal of the local ring $R/\mathrm{Ann}\,(M)$ is a finite sum of minimal ideals. Thus $P/\mathrm{Ann}\,(M)$ has this property, for P is finitely generated and $P^2\subseteqq\mathrm{Ann}\,(M)$ so that $P/\mathrm{Ann}\,(M)$ is a finite-dimensional vector space over the field R/P. Since each proper ideal of $R/\mathrm{Ann}\,(M)$ is contained in $P/\mathrm{Ann}\,(M)$, it follows that each proper ideal of $R/\mathrm{Ann}\,(M)$ is a finite sum of minimal ideals, and our proof of Theorem 1 is complete.

We observe that if R is a local ring with nonzero maximal ideal P such that $P^2=(0)$, then each proper submodule of the R-module R is a finite sum of simple submodules, while R itself is not completely reducible; moreover, both P and R/P are finite sums of simple modules in this case. Thus the answer to the question posed at the beginning of the paper is negative. We conclude with a positive result concerning completely reducible modules.

Proposition 2. *Assume that the R-module M is the sum of a finite family $\{M_i\}_1^n$ of submodules. If each M_i is completely reducible, then so is M.*

Before embarking upon the proof, we establish a lemma that allows us to reduce to the case where M is the direct sum of the family $\{M_i\}_1^n$.

Lemma 1. *If $\{M_i\}_1^n$ is a finite family of completely reducible submodules of an R-module M, then $\sum_1^n M_i$ is the direct sum of a finite family of completely reducible submodules.*

PROOF. It suffices to consider the case $n=2$. The submodule $M_1\cap M_2$ is a direct summand of M_1, say $M_1=N_1\oplus(M_1\cap M_2)$. It then follows that $M_1+M_2=$ $=N_1\oplus M_2$, where N_1, as a submodule of M_1, is completely reducible.

PROOF OF PROPOSITION 2. By Lemma 1 and induction, it suffices to show that each submodule N of $M_1\oplus M_2$ is a direct summand of $M_1\oplus M_2$. We consider first the case where $N\cap M_1=N\cap M_2=(0)$. If ϱ is the projection map of $M_1\oplus M_2$ onto M_1, then $\varrho(N)$ is a direct summand of M_1 — say $M_1=\varrho(N)\oplus U$; we prove that $M_1\oplus M_2=N\oplus U\oplus M_2$. If $m_1\in M_1$ and if $m_1=\varrho(n)+u$ is the decomposition of m_1 with respect to the direct decomposition $\varrho(N)\oplus U$ of M_1, then $m_1=$ $=n+u_1+(\varrho(n)-n)$, where $\varrho(n)-n\in M_2$, and hence $M_1\oplus M_2=N+U+M_2$. To show that the sum $N+U+M_2$ is direct, we need only prove that $N\cap(U+M_2)=$ $=(0)$. Thus if $n=u+m_2$ is in $N\cap(U+M_2)$, where $u\in U$ and $m_2\in M_2$, then $\varrho(n)=$ $=u\in\varrho(N)\cap U=(0)$ and $n=m_2\in N\cap M_2=(0)$.

In the general case, $N\cap M_i$ is a direct summand of M_i, say $M_i=(N\cap M_i)\oplus N_i$. Then $M_1\oplus M_2=(N\cap M_1)\oplus(N\cap M_2)\oplus N_1\oplus N_2$ and $N=(N\cap M_1)\oplus(N\cap M_2)$ $\oplus[N\cap(N_1\oplus N_2)]$. The modules N_i are completely reducible and $N\cap(N_1\oplus N_2)$ is such that $[N\cap(N_1\oplus N_2)]\cap N_i=N\cap N_i=N\cap(N_i\cap M_i)=(N\cap M_i)\cap N_i=(0)$. By the case previously considered, $N_1\oplus N_2=[N\cap(N_1\oplus N_2)]\oplus N_3$ for some submodule N_3. Therefore $M_1\oplus M_2=N\oplus N_3$, and this completes the proof of Proposition 2.

References

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