On radicals in a certain class of semigroups

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We shall call a semigroup S a σ-reflexive semigroup if any subsemigroup H in S is reflexive (that is, for all a, b in S, $ab\in H \Rightarrow ba\in H$) ([2, 3, 7]). A commutative semigroup is clearly a σ-reflexive semigroup. J. Bosák [1, p. 209] and R. Šulka [6, p. 221] proved that if S is a commutative semigroup and J any ideal of S, then the Clifford, McCoy, Ševrin, Schwarz and Luh radicals with respect to J (respectively denoted by $R_J^*(S), M_J(S), L_J(S), R_J(S)$ and $C_J(S)$) are equal to $N_J(S)$, the set of all nilpotent elements of S with respect to J. For their definitions, we refer to [1, 6]. Further J. E. Kuczkowski [4] proved that if S is a C_2-semigroup, then $M_J(S)=L_J(S)=R_J^*(S)=C_J(S)=N_J(S)$ for every ideal J of S. The purpose of this note is to extend the previous result to the class of σ-reflexive semigroups obtaining the following

Theorem. *If S is a σ-reflexive semigroup and J any ideal of S; then* $R_J(S)=$ $=M_J(S)=L_J(S)=R_J^*(S)=N_J(S)=C_J(S)$.

First we recall the following *proposition from* [2].

Proposition. Any semigroup S is σ-reflexive if and only if it satisfies the following condition:

$$\forall\, a, b\in S, \quad \exists m = m(a, b) \geqq 1; \quad ab = (ba)^m.$$

Before coming to the proof of the theorem, we prove the following two lemmas.

Lemma 1. *Let S be a σ-reflexive semigroup. Then an ideal of S is prime if and anly if it is completely prime* [1].

Proof. Clearly it suffices to prove that any prime ideal is completely prime. Let P be any prime ideal and $ab\in P$, where $a, b\in S$. We shall show that $a\in P$ or $b\in P$. Let x be any arbitrary element of S. Then by the previous proposition, $ax=(xa)^m$, where m is an integer $\geqq 1$. Now $axb=(xa)^m$. $b\in P$ for every x in S, since $ab\in P$, whence $aSP\subseteq P$. As P is prime, we get $a\in P$ or $b\in P$, so that P is a completely prime ideal.

Corollary. Let S be a σ-reflexive semigroup and J any ideal of S. Then $M_J(S)=C_J(S)$.

Let x be any element of a semigroup S. The principal ideal generated by x will be denoted by $J(x)$.

Lemma 2. *Let S be a σ-reflexive semigroup. Then for any x, y in S, $J(x)\cdot J(y)=$ $=J(xy)$.*

Proof. Clearly $J(xy)\subseteqq J(x)\cdot J(y)$. Let $a\in J(x)$ and $b\in J(y)$. Then a is one of $x, t_1 x, xt_2$, or $t_1 xt_2$ and b is one of $y, t_3 y, yt_4$, or $t_3 yt_4$ where $t_i\in S$ for $i=$ $=1, 2, 3, 4$. By making use of the previous proposition, it is a routine verification that $ab\in J(xy)$ in every case. Thus $J(x)\cdot J(y)\subseteqq J(xy)$, whence the lemma.

Proof of the Theorem. J. Bosák [1, Theorem 2] proved that

$$R_J(S)\subseteqq M_J(S)\subseteqq L_J(S)\subseteqq R_J^*(S)\subseteqq N_J(S)\subseteqq C_J(S),$$

for any semigroup S. Since $M_J(S)=C_J(S)$ by the above corollary, we get the equality of five of them barring $R_J(S)$. This will be equal to the others if we show $R_J^*(S)\subseteqq R_J(S)$ and this we do now. Let $a\in R_J^*(S)$; then $a^n\in J$ for some positive integer n. Since $[J(a)]^n=J(a^n)\subseteqq J$, by Lemma 2, $J(a)$ is a nilpotent ideal with respect to J and hence $a\in R_J(S)$. This completes the proof of the theorem.

Remark. This theorem no more remains true if we drop from its hypothesis the σ-reflexivity of the semigroup. For instance, let $S=\{0, a_1, a_2, a_3, a_4\}$ be a semigroup with the following multiplication table:

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	0	a_3	0
a_2	0	0	a_2	0	a_4
a_3	0	0	a_3	0	a_1
a_4	0	a_4	0	a_2	0

Here $H=\{0, a_2, a_4\}$ is a subsemigroup of S such that $a_3 a_1(=0)$ is in H, whereas $a_1 a_3(=a_3)$ is not in H; whence S is not σ-reflexive. Let $J=\{0\}$; then $R_J(S)=$ $=M_J(S)=L_J(S)=R_J^*(S)=\{0\}$; $N_J(S)=\{0, a_3, a_4\}$ and $C_J(S)=S$. This example is due to Š. Schwarz [5].

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