Tridiagonal matrices and functions analytic in two half-planes

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The problem of eigenvalues and eigenvectors of real symmetric tridiagonal matrices is known to be equivalent to a second order discrete boundary problem. The characteristic polynomials of these matrices are orthogonal in a certain sense. The best way to derive some of their properties arises while studying the above mentioned boundary problem. The aim of this paper is to show that in case of antisymmetric[1]) tridiagonal matrices similar methods with some modifications may give further interesting results.

I. Introduction

The second order discrete boundary problem may generally be formulated for the difference equation as:

(1) $$c_n y_{n+1} = (a_n + \lambda b_n) y_n - c_{n-1} y_{n-1},$$

where n are non-negative integers, a_n, b_n, c_n are real numbers, $b_n > 0$, $c_n > 0$. Let to be found a nontrivial set of constants $y_0, y_1, \dots y_m$ such that $y_m + h y_{m-1} = 0$ for a fixed real number h and a fixed integer m while taking $y_{-1} = 0$.

If considering $y_k = y_k(\lambda)$ it can be seen that y_k are polynomials in the λ variable and the eigenvalues of the boundary problem are the zeros of the polynomial $y_m(\lambda) + h y_{m-1}(\lambda)$.

Let A denote the tridiagonal matrix of order $m+1$ with elements

$$d_{ik} = \begin{cases} a_{k-1}/b_{k-1} & \text{for} \quad i = k \\ c_{k-1}/\sqrt{b_{k-1} b_k} & \text{for} \quad i-1 = k \\ c_{i-1}/\sqrt{b_{i-1} b_i} & \text{for} \quad i+1 = k \\ 0 & \text{otherwise} \end{cases}$$

Its eigenvalues coincide with those of the above formulated boundary problem for $h=0$; the coordinates of its eigenvectors are the above mentioned constants $y_0, y_1, \dots \dots y_m$.

The results concerning the eigenvalues and eigenvectors of the formulated boundary problem of the matrix A can be found in the basic literature (see e.g. [1],

[1]) A matrix A is called antisymmetric if its sum with its transpose equals to a diagonal matrix (not necessarily zero).

[2], [3]). Besides these two initial formulations an additional possibility is being applied: the polynomials $y_k(\lambda)$ are orthogonal polynomials (e.g. $c_n=1, a_n=0, b_n=2$ gives the Hermite polynomials, e.t.c.). The elementary results of their unified theory are generally known: all the zeros of these polynomials (the eigenvalues of the corresponding boundary problem or the eigenvalues of the corresponding tridiagonal matrix) are real numbers, they are simple roots of the corrensponding polynomials with so called "interlacing" property.

Let us investigate a modified second order discrete boundary problem leading to an antisymmetric tridiagonal matrix. It can be shown that many consequences of the outlined theory remain valid, although none of the simple properties of eigenvalues as mentioned previously can be proved. This modified boundary problem has many interesting relations with theories such as the theory of analytic functions and their integral representations, the theory of continued fractions, the theory of moments etc. similarly as in the "classical" case. In the latter case all these relationships are treated in the three outstanding books mentioned before. The methods given in these books will be followed in this paper.

II. A difference equation and the corresponding boundary problem; basic results

Let us consider the following second order difference equation:

$$y_{k+1}-y_{k-1}=(\alpha_k+\lambda\beta_k)y_k \tag{2}$$

where α_k, β_k are real constants, $\beta_k\neq 0, k=0, 1, \ldots$. Any sequence $\{y_k(\lambda)\}_{k=0}^{\infty}$ satisfying (2) will be called its solution belonging to λ. Any $(m+1)$-dimensional vector $\{y_0(\lambda), y_1(\lambda), \ldots, y_k(\lambda)\}$ will be called its solution of order m belonging to λ; $y_k(\lambda)$ are coordinates of this solution. For standard initial values we shall use the notation as follows: $P_k(\lambda)$ stands for the *k-th* coordinate if $y_{-1}=P_{-1}=0$ and $y_0=P_0=1$; similarly $Q_k(\lambda)$ if $y_{-1}=Q_{-1}=1$ and $y_0=Q_0=0$.

In general if y_{-1} and y_0 are constants, $y_0\neq 0$, all the coordinates $y_k(\lambda)$ are polynomials of exact degree k; if $y_0=0$ then deg $y_k<k$.

To compare equation (2) with (1) we may replace c_n in equ. (1) by $1/d_n d_{n+1}$ and denote $y_k/d_k=Y_k$; we obtain

$$Y_{n+1}+Y_{n-1}=(a_n d_n^2+b_n d_n^2\lambda)Y_n. \tag{1'}$$

This equation shows the distinction between (1) and (2).

The following problems can be formulated:

A: Let A be an antisymmetric tridiagonal matrix of order m with off — diagonal elements $a_{i,i-1}=-a_{i-1,i}\neq 0$. The eigenvalues and eigenvectors are to be found.

B: Let equ. (2) be given. All the m dimensional vectors $\{y_0, y_1, \ldots y_{m-1}\}$ satisfying (2) and the boundary conditions $y_{-1}=y_m=0$ are to be found.

C: Let a finite continued fraction be given as:

$$f_m(\lambda)=\frac{1}{\beta_{m-1}\lambda+\alpha_{m-1}}+\frac{1}{\beta_{m-2}\lambda+\alpha_{m-2}}+\cdots+\frac{1}{\beta_0\lambda+\alpha_0} \tag{3}$$

with $\beta_i \neq 0, i=0, 1, \dots m-1$. All the poles and zeros of the rational function f_m are to be found.

Denoting in problem A the elements $a_{ii} = -\alpha_{i-1}/\beta_{i-1}$ and $a_{i+1,i} = -1/\sqrt{\beta_{i-1}\beta_i}$ (i.e. supposing without loss of generality that all the subdiagonal elements are of equal sign) it can be shown, that the problems A, B, C are essentially equivalent: the solution of any of them gives in fact the solution of the remaining two.

Now let us formulate a few results concerning solutions of equation (2).

It will be useful to note that the coordinates of the "standard" solution, the polynomials P_k are identical with the determinants

$$(4) \qquad P_k(\lambda) = \begin{vmatrix} \alpha_0+\beta_0\lambda & -1 & 0 & \dots & 0 & 0 \\ 1 & \alpha_1+\beta_1\lambda & -1 & \dots & 0 & 0 \\ 0 & 1 & \alpha_2+\beta_2\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{n-2}+\beta_{n-2}\lambda & -1 \\ 0 & 0 & 0 & \dots & 1 & \alpha_{n-1}+\beta_{n-1}\lambda \end{vmatrix}$$

for $k \geqq 1$.

1. Proposition. *Let* $\{x_k\}$ *and* $\{y_k\}$ *be any solutions of equ.* (2) *belonging to* μ *and* λ *respectively. Then for any positive integer* k *there is*

$$(5) \qquad \begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix} = (-1)^k (x_0 y_{-1} - x_{-1} y_0) + (\lambda-\mu) \sum_{i=0}^{k} (-1)^{i+k} \beta_i x_i y_i$$

and

$$(6) \qquad \begin{vmatrix} x_k & -x_{k+1} \\ y_k & y_{k+1} \end{vmatrix} = x_{-1} y_0 + x_0 y_{-1} + \sum_{i=0}^{k} [2\alpha_i + \beta_i(\mu+\lambda)] x_i y_i .$$

Proof. Let the left-hand side of (5) be M_k. There is

$$M_i + M_{i-1} = \begin{vmatrix} x_i & x_{i+1}-x_{i-1} \\ y_i & y_{i+1}-y_{i-1} \end{vmatrix} = x_i y_i \begin{vmatrix} 1 & \alpha_i+\beta_i\mu \\ 1 & \alpha_i+\beta_i\lambda \end{vmatrix} = (\lambda-\mu)\beta_i x_i y_i .$$

Multiplying each of these equations for $i=0, 1, \dots$ with $(-1)^i$ we obtain the first of the desired results by summation. Similarly, forming $N_i - N_{i-1}$ in (6) we get the second formula.

Let us put $\bar{\mu} = \lambda = z$ and apply the relation $\bar{y}_k(\lambda) = y_k(\bar{\lambda})$. From (5) and (6) follows:

$$(7) \qquad \operatorname{Re} \bar{y}_k y_{k+1} = \operatorname{Re} \bar{y}_0 y_{-1} + \sum_{i=0}^{k} (\alpha_i + \beta_i \operatorname{Re} z) |y_i(z)|^2$$

and

$$(8) \qquad \operatorname{Im} \bar{y}_k y_{k+1} = (-1)^k \operatorname{Im} \bar{y}_0 y_{-1} + \operatorname{Im} z \sum_{i=0}^{k} (-1)^{i+k} \beta_i |y_i(z)|^2 .$$

In equation (5) we may subtract on the left side the first row from the second one; after dividing equ. (5) by $(\lambda-\mu)$ we obtain:

If $x_0y_{-1}-x_{-1}y_0=0$ for any μ and λ, in particular for any standard solution P_k or Q_k, there is (the prime stands for $d/d\lambda$)

$$\begin{vmatrix} x_k(\lambda) & x_{k+1}(\lambda) \\ x_k'(\lambda) & x_{k+1}'(\lambda) \end{vmatrix} = \sum_{i=0}^{k}(-1)^{i+k}\beta_i x_i^2(\lambda). \tag{9}$$

From (3) we get that any two linearly independent solutions p_k and q_k of (2), both belonging to λ satisfy identically the equation.

$$p_k q_{k+1}-q_k p_{k+1} = (-1)^k(p_0 q_{-1}-p_{-1}q_0) \tag{10}$$

and in particular the standard solutions satisfy

$$P_k Q_{k+1}+Q_k P_{k+1} = (-1)^k. \tag{11}$$

The proof of the following proposition immediatelly comes from (10):

2. Proposition. *Two consecutive coordinates of any solution, belonging to λ cannot have any common zero; the same holds true for any corresponding two coordinates of two independent solutions of the equation* (2).

3. Proposition. *Let $\beta_i>0$ for $i\leqq k$ and let the initial conditions of equ.* (2) *satisfy the condition* Re $\bar{y}_0 y_{-1}=0$. *Then all the zeros of the k-th coordinate of the solution are contained in a strip parallel to the imaginary axis:*

$$y_k(\lambda_0)=0 \Rightarrow \min_{i\leqq k}(-\alpha_i/\beta_i) \leqq \operatorname{Re}\lambda_0 \leqq \max_{i\leqq k}(-\alpha_i/\beta_i).$$

The proof follows from (7); for any λ outside of the strip we have either $\alpha_i+\beta_i \operatorname{Re}\lambda>0$ or $\alpha_i+\beta_i \operatorname{Re}\lambda<0$ for all $i=0, 1, \ldots, k$ and therefore $\bar{y}_k y_{k+1}+ +y_k\bar{y}_{k+1}\neq 0$.

4. Proposition. *Let $y_k(\lambda_0)=0$; λ_0 is a multiple zero of $y_k(\lambda)$ if and only if*

$$\sum_{i=0}^{k-1}(-1)^i\beta_i y_i^2(\lambda_0)=0. \tag{12}$$

Proof. The "if statement" follows from (9) while considering 2. Proposition the converse is evident from (9).

The equivalence of Problems B and C can be proved:

5. Proposition. *The ratio of two consecutive coordinates of the standard solution P_k of equ.* (2) *equals to the continued fraction* (3) *i.e. $f_m(z)=P_{m-1}(z)/P_m(z)$.*

Proof. Evidently $f_1(z)=1/(\alpha_0+\beta_0 z)$. Now the proof comes by induction. Let $f_k(z)=P_{k-1}(z)/P_k(z)$. Take $P_{k-1}(z)/P_k(z)+\alpha_k+\beta_k z$. Evidently its inverse equals $f_{k+1}(z)$; we get

$$f_{k+1}(z) = P_k(z)/[\alpha_k+\beta_k z)P_k(z)+P_{k-1}(z)] = P_k(z)/P_{k+1}(z),$$

which completes the proof.

It can be verified that

$$\frac{Q_m(z)}{P_m(z)} = \frac{1}{\alpha_0+\beta_0 z}+\frac{1}{\alpha_1+\beta_1 z}+\ldots+\frac{1}{\alpha_{m-1}+\beta_{m-1}z}. \tag{13}$$

When, in addition to what has been said $P_m(z)$ is considered as in formula (4), it can be concluded that the equivalence of problems A, B, C has been established.

This paragraph can be finished with a proposition, that motivates the considerations below.

6. Proposition. *Let f_m be as in 5. Proposition and let $D(\gamma, \Gamma)$ be the complement of the strip in 3. Proposition;* $\gamma=\min_i\left(-\frac{\alpha_i}{\beta_i}\right)$, $\Gamma=\max_i\left(-\frac{\alpha_i}{\beta_i}\right)$. *Then f_m is a function analytic and nonzero in both simply connected components of the domain $D(\gamma, \Gamma)$. Moreover, f_m maps the half-planes* $\operatorname{Re} z>\Gamma$ *and* $\operatorname{Re} z<\gamma$ *respectively onto the right and left half-planes.*

PROOF. The first part is an evident consequence of 3. Proposition. The second statement follows from equ. (7).

7. *Corollary.* Statements of 6. Proposition hold true for continued fractions with terms $1/(\alpha_i+\beta_i z)$, $i=0, \ldots, m-1$ in arbitrary order.

8. *Corollary.* The eigenvalues λ_k of a tridiagonal antisymmetric matrix $A=[a_{ik}]$, $a_{i+1,i}=-a_{i,i+1}>0$, $a_{ik}=0$ for $|i-k|>1$, $i, k=1, 2, \ldots, n$ satisfy the inequality

$$\min_{1\leqq i\leqq n} a_{ii} \leqq \operatorname{Re}\lambda_k \leqq \max_{1\leqq i\leqq n} a_{ii}.$$

III. Functions analytic in two half-planes

The results given above, namely the 6. Proposition shows that it could be useful to introduce a special class of analytic functions. The following notations will be used: Let γ, Γ be real numbers, $\gamma\leqq\Gamma$, with cases $\gamma=-\infty$ or $\Gamma=+\infty$ not excluded. The set of complex numbers z with $\operatorname{Re} z>\alpha$ and $\operatorname{Re} z<\alpha$ will be denoted by D_+^α and D_-^α respectively; let for $\gamma\leqq\Gamma$ be $D=D(\gamma, \Gamma)=D_-^\gamma\cup D_+^\Gamma$; evidently the strip in 3. or 6. Propositions is equal to the set CD (C for the complement).

9. *Definition.* Let $G(\gamma, \Gamma)$ denote the class of functions f with the following properties:

i) f is a function analytic in $D(\gamma, \Gamma)$,

ii) $f(\bar{z})=\overline{f(z)}$ for $z\in D(\gamma, \Gamma)$,

iii) $f(D_-^\gamma)\subset D_-^0$, $f(D_+^\Gamma)\subset D_+^0$,

iv) f in D_-^γ and f in D_+^Γ are the analytic continuation of each other.

Note that the class $G(-\infty, 0)$ is identical with the class of positive real functions. Some results and methods relating to this class may be used here (see [6], [7],). The following proposition illustrates this:

10. Proposition. *Let $f\in G(\gamma, \Gamma)$. Then the function f cannot have any zero in the domain $D(\gamma, \Gamma)$ and all its zeros z_0 with* $\operatorname{Re} z_0=\Gamma$ *or* $\operatorname{Re} z_0=\gamma$ *must be simple.*

PROOF. Let $z_0 \in D(\gamma, \Gamma)$ and $f(z_0)=0$. Then, in a sufficiently small neighbourhood of z_0 we have

$$f(z) = r^k e^{jk\varphi}(a_k + a_{k+1} r e^{j\varphi} + \dots), \quad a_i \text{ real},$$

where $z = z_0 + re^{j\varphi}$, $a_k \neq 0$, k equals to the order of the considered zero-point. The real part of the first term here $\operatorname{Re} a_k r^k e^{jk\varphi}$ assumes positive as well as negative values for $k \geqq 1$ and $-\pi < \varphi \leqq \pi$ contrary to our hypothesis that $f \in G(\gamma, \Gamma)$. Let now be e.g. $\operatorname{Re} z_0 = \Gamma$ (the case $\operatorname{Re} z_0 = \gamma$ can be treated similarly). Then $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ and the above argument holds true for $k \geqq 2$, therefore it must be $k \leqq 1$.

The following simple lemma summarizes some conclusions concerning the class $G(\gamma, \Gamma)$; the lemma needs no further proof if noting that $\operatorname{Re} z \operatorname{Re} \frac{1}{z} > 0$ for any complex number z with $\operatorname{Re} z \neq 0$ and if considering 10. Proposition.

11. Lemma. *The following statements are valid:*

i) *if* $\gamma' \leqq \gamma$ *and* $\Gamma' \geqq \Gamma$ *then* $G(\gamma, \Gamma) \subset G(\gamma', \Gamma')$;

ii) $f \in G(\gamma, \Gamma)$ *if and only if* $1/f \in G(\gamma, \Gamma)$;

iii) $f \in G(\gamma, \Gamma)$ *if and only if for any* $a > 0$ *there is* $af \in G(\gamma, \Gamma)$;

iv) *let* α, β *be nonnegative numbers,* $f \in G(\gamma_1, \Gamma_1)$, $g \in G(\gamma_2, \Gamma_2)$, *then* $\alpha f + \beta g \in G(\min(\gamma_1, \gamma_2), \max(\Gamma_1, \Gamma_2))$.

12. Lemma. *Let* $\varphi \in G(\gamma, \Gamma)$ *and* $a_0 > 0$; *then for any real* $\frac{a_1}{a_0}$

$$f(z) = \frac{a_0}{z - \frac{a_1}{a_0} + \varphi(z)} \in G\left(\min\left(\frac{a_1}{a_0}, \gamma\right), \max\left(\frac{a_1}{a_0}, \Gamma\right)\right). \tag{14}$$

The proof follows immediately from 11. Lemma when considering that

$$z - \frac{a_1}{a_0} \in G\left(\frac{a_1}{a_0}, \frac{a_1}{a_0}\right).$$

13. Propositon. *Let* γ *and* Γ *be finite and* $f \in G(\gamma, \Gamma)$. *Then there exist two nondecreasing functions* τ_γ *and* τ_Γ, *both with their even part equivalent to zero and with*

$$\int_0^\infty \frac{d\tau_i(t)}{1+t^2} < +\infty \quad \text{for} \quad i = \gamma, \Gamma$$

and two nonnegative numbers A_i, $i = \gamma, \Gamma$ *such that*

$$f(z) = \begin{cases} A_\Gamma(z-\Gamma) + \int_{-\infty}^{+\infty} \frac{d\tau_\Gamma}{z-\Gamma+jt} & \text{for} \quad z \in D_+^\Gamma \\ A_\gamma(z-\gamma) + \int_{-\infty}^{+\infty} \frac{d\tau_\gamma}{z-\gamma+jt} & \text{for} \quad z \in D_-^\gamma \end{cases} \tag{15}$$

here, $\int_{-\infty}^{+\infty}$ *means the "valeur principale",* $j^2 = -1$.

PROOF. It has been proved earlier [4], that $\varphi \in G(-\infty, 0)$ if and only if there exists a nonnegative number μ and a nondecreasing function τ with its even part equivalent to zero and satisfying

$$\int_0^{\infty} \frac{d\tau(t)}{1+t^2} < +\infty \quad \text{such that} \quad \varphi(z) = \mu z + \int_{-\infty}^{+\infty} \frac{d\tau(t)}{z+jt} \quad \text{for} \quad z \in D_+^0 .$$

Let us take $f \in G(\gamma, \Gamma)$. Since condition $\operatorname{Re} z > \Gamma$ implies $\operatorname{Re} f(z) > 0$, it follows for $w = z - \Gamma$: $\operatorname{Re} w > 0 \Rightarrow \operatorname{Re} f(z) > 0$. Therefore

$$f(z) = A_\Gamma (z - \Gamma) + \int_{-\infty}^{+\infty} \frac{d\tau_\Gamma(t)}{z - \Gamma + jt}$$

because all other assumptions are evidently satisfied.

The citied result can be modified to be applied in case $\psi \in G(0, +\infty)$. Namely, $\psi \in G(0, +\infty) \Leftrightarrow$

$$\psi(z) = \nu z + \int_{-\infty}^{+\infty} \frac{d\tau(t)}{z+jt} \quad \text{for} \quad z \in D_-^0 .$$

with $\nu \geqq 0$ and τ with the same properties as above. Therefore with similar reasoning for the half-plane $\operatorname{Re} z < \gamma$ the second part of our statement holds true. It makes the proof complete.

The functions τ_Γ and τ_γ can be expressed by the Stieltjes—Perron formula [1] or by its modification [4]. Without loss of generality it can be assumed that $\tau_i(0) = 0$, $i = \gamma, \Gamma$. For any real t and c denoting

$$\tau_i(t;\, c) = \frac{1}{2}\big(\tau_i(t+0) + \tau_i(t-0)\big) + \frac{1}{2}\big(\tau_i(c+0) + \tau_i(c-0)\big)$$

the functions τ_i are given as

$$\tau_\Gamma(t;\, c) = \lim_{x \to \Gamma_+} \frac{1}{\pi} \int_c^t \operatorname{Re} f(x+jy)\, dy,$$

(16)

$$\tau_\gamma(t;\, c) = -\lim_{x \to \gamma_-} \frac{1}{\pi} \int_c^t \operatorname{Re} f(x+jy)\, dy.$$

Using once more the results in [4] and iv) from 9. Definition it can be stated:

14. Lemma. *Let* $f \in G(\gamma, \Gamma)$ *and* $\lim\limits_{z \to \infty} \dfrac{f(z)}{z} = B \neq \infty$ *on any closed subset of* $D(\gamma, \Gamma)$. *Then* $B = A_\gamma = A_\Gamma$.

The relationship between the class $G(\gamma, \Gamma)$ and the continued fractions of certain type has already been shown in 12. Lemma. In this direction a stronger result can be formulated:

15. Proposition. *Let* $f \in G(\gamma, \Gamma)$, *and let* $\lim_{z\to\infty} \frac{1}{zf(z)} = A$, A, $\gamma < \Gamma$ *finite. If there exists a real number* α *satisfying the condition*

$$A(\Gamma-\gamma) \leqq \alpha \leqq \operatorname{Re} \frac{1}{f(\Gamma+jt)}$$

for all real values of t, *then the function* φ *defined by*

$$f(z) = \frac{1}{A(z-\Gamma)+\alpha+\varphi(z)}$$

belongs to the class $G(\gamma, \Gamma)$. *Similarly, if there exists a real number* β *such that*

$$A(\gamma-\Gamma) \leqq -\beta \leqq \operatorname{Re} \frac{1}{f(\gamma+jt)}$$

then the function ψ *defined by*

$$f(z) = \frac{1}{A(z-\gamma)-\beta+\psi(z)}$$

belongs to the calss $G(\gamma, \Gamma)$.

If, moreover the number α *(or* $-\beta$*) is the exact bound of*

$$\operatorname{Re} \frac{1}{f(\Gamma+jt)} \quad \left(\text{or } \operatorname{Re} \frac{1}{f(\gamma+jt)}\right) \quad \text{then} \quad \lim_{z\to\infty} \varphi(z) = 0 \quad (\lim_{z\to\infty} \psi(z) = 0).$$

Proof. We shall prove the statement concerning only the function φ. The proof of the rest could be given almost identically. From $f \in G(\gamma, \Gamma)$ follows that $\frac{1}{f} \in G(\gamma, \Gamma)$ and

$$\frac{1}{f(z)} = A(z-\Gamma) + \int_{-\infty}^{+\infty} \frac{d\tau_\Gamma}{z-\Gamma+jt}$$

for all z satisfying $\operatorname{Re} z > \Gamma$. Here,

$$\tau_\Gamma = \frac{1}{\pi} \lim_{x\to\Gamma_+} \int_c^t \operatorname{Re} \frac{1}{f(x-jy)}\, dy.$$

Evidently, the function

$$\tau_\Gamma^* = \frac{1}{\pi} \lim_{x\to\Gamma_+} \int_c^t \left(\operatorname{Re} \frac{1}{f(x+jy)} - \alpha\right) dy$$

satisfies all the relevant conditions of 13. Proposition and therefore the function

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{d\tau_\Gamma^*}{z-\Gamma+jt}$$

has the following properties: i) $\varphi(D_+^\Gamma)\subset D_+^0$, ii) $\varphi(z)=\frac{1}{f(z)}-A(z-\Gamma)-\alpha$. We may use ii) for the analytic continuation of the function φ. Using again the assumption $f\in G(\gamma,\Gamma)$, we get for $\operatorname{Re} z<\gamma$

$$\varphi(z)=A(z-\gamma)+\int_{-\infty}^{+\infty}\frac{d\tau_\gamma}{z-\gamma+jt}-A(z-\Gamma)-\alpha=A(\Gamma-\gamma)-\alpha+\int_{-\infty}^{+\infty}\frac{d\tau_\gamma}{z-\gamma+jt}.$$

The constant $A(\Gamma-\gamma)-\alpha$ is nonpositive and so is the real part of the integral for $\operatorname{Re} z<\gamma$; therefore $\varphi(D_-^\gamma)\subset D_-^0$ and together with the property i) above this means that $\varphi\in G(\gamma,\Gamma)$.

Let now be $\alpha=\lim_{t\to\infty}\operatorname{Re}\frac{1}{f(\Gamma+jt)}$. By the appropriate choice of B can the modulus of the integral $\int_{|t|>B}\frac{d\tau_\Gamma^*}{z-\Gamma+jt}$ be made arbitrarily small for any value of z. In the half-plane $\operatorname{Re} z>\Gamma$ can also the value of $\left|\int_{-B}^{B}\frac{d\tau_\Gamma^*}{z-\Gamma+jt}\right|$ be made arbitrarily small when choosing $|z|$ large. This completes the proof.

If $f\in G(\gamma,\Gamma)$ with $\gamma=\Gamma$, the above proposition can be simplified:

16. Proposition. *Let* $f\in G(\Gamma,\Gamma)$. *Then*

$$f(z)=\frac{1}{A(z-\Gamma)+\varphi(z)},\quad A\geqq 0$$

where $\varphi(z)\in G(\Gamma,\Gamma)$ *and* $\lim_{z\to\infty}\frac{\varphi(z)}{z}=0$.

The proof proceeds in the same manner as before though it is much simpler.

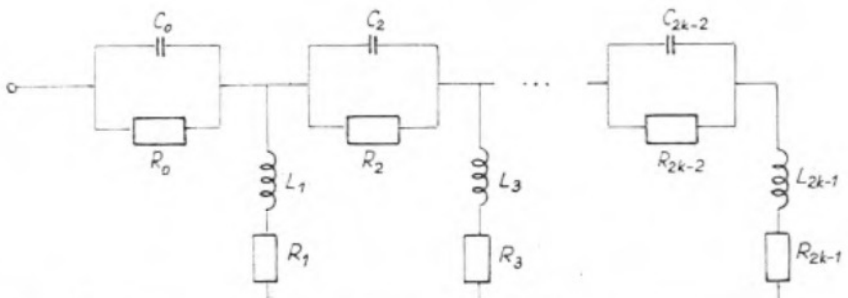

Fig. 1

IV. Examples

17. *Example.* The driving-point impedance of a series-parallely network consisting of constant resistances, inductances and capacitances as shown in Fig. 1. can be expressed as a finite continued fraction

$$f(z) = \frac{1}{R_0^{-1}+C_0 z} + \frac{1}{R_1+L_1 z} + \cdots + \frac{1}{R_{2k-1}+L_{2k-1} z}.$$

Therefore $f \in G(\gamma, \Gamma)$, where

$$\gamma = \min(-R_i^{(-1)^{i+1}}/\varkappa_i), \quad \Gamma = \max(-R_i^{(-1)^{i+1}}/\varkappa_i)$$

$$\varkappa_i = \begin{cases} C_{i/2} & \text{for } i \text{ even} \\ L_{(i+1)/2} & \text{for } i \text{ odd.} \end{cases}$$

Essentially the same functions f with different meaning of constants have been used by MITRA and SAGAR [5] as transfer functions of digital filters.

18. *Example.* For approximation purposes in network theory a set of polynomials ω_k is being used. These polynomials satisfy the equation

$$\omega_k\left(\frac{1}{s}\right) = \frac{2k-1}{s}\,\omega_{k-1}\left(\frac{1}{s}\right) + \omega_{k-2}\left(\frac{1}{s}\right),$$

which may readily be identified with equation (2) for $\alpha_k=0, \beta_k=2k+1, \lambda=1/s$. Solutions of this equation with standard initial conditions are known as Lommel polynomials. In the network theory the so-called Bessel polynomials are used; these satisfy the above equation and the initial conditions $w_{-1}=w_0=1$, which means that $w_k=P_k+Q_k$. From equ. (7) follows: all the zeros of Lommel polynomials are simple and lie on the imaginary axis. The Bessel polynomials are stricly Hurwitz polynomials, the ratio of the two consecutive Bessel polynomials is a positive real function.

The Lommel and Bessel polynomials together with some of their applications can be found in [7] in more detail.

The boundary problem (2) and the class $G(\gamma, \Gamma)$ of the functions are closely related to many interesting topics which could not be dealt with, here. Some further results on infinite continued fractions and the problem of moments will be published later.

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