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On groups with a locally nilpotent triple factorization

By BERNHARD AMBERG (Mainz) and YAROSLAV SYSAK (Kiev)

Abstract. The following theorem is proved. Let the group $G = AB = AM = BM$ be the product of three locally nilpotent subgroups A, B and M, where M is normal in G. If M has an ascending G-invariant series with minimax factors, then G is locally nilpotent.

1. Introduction

In the theory of groups which have a factorization, groups of the form $G = AM = BM = AB$ with two subgroups A and B and a normal subgroup M of G play a special role. A general construction of such "triply factorized" groups is due to the second author and can be found in Section 6.1 of [2]. There are many situations in which the triply factorized group G satisfies some nilpotency condition if the three subgroups A, B and M satisfy this nilpotency condition. For instance, if M is a minimax group or if G has finite abelian section rank, then it was shown in [1] that the local nilpotency of A, B and M implies that of G (see [2], Theorems 6.3.7 and 6.3.8). The following theorem extends these results.

Theorem 1.1. *Let the group $G = AB = AM = BM$ be the product of three locally nilpotent subgroups A, B and M, where M is normal in G. If M has an ascending G-invariant series with minimax factors, then G is locally nilpotent.*

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Observe that in Theorem [1] the normal subgroup M of G is hypercentrally embedded in G. Therefore, if the subgroups A, B and M are hypercentral, then also the group G is hypercentral. On the other hand, there exist non-nilpotent groups which have a triple factorization with three abelian factors, these groups are even hypercentral with Prüfer rank 2 (see [2], Example 6.3.5). It should also be noted that Theorem 1.1 cannot be extended to the case when M has an ascending G-invariant series whose factors have finite Prüfer rank. This can be seen from the example of a triply factorized group $G = AB = AM = BM$ in [2], Theorem 6.1.2, where the subgroups A, B and M are abelian and M has Prüfer rank 1, but G is not locally nilpotent.

Recall that the FC-central series $F_\alpha(G)$ is defined by the rules $F_0(G) = 1$, $F_{\alpha+1}(G)/F_\alpha(G)$ is the FC-centre of $G/F_\alpha(G)$ and $F_\lambda(G) = \bigcup F_\beta(G)$ $(\beta < \lambda)$ where α is an ordinal and λ is a limit ordinal. The group G is FC-hypercentral if $F_\alpha = G$ for some ordinal α, and it is FC-nilpotent if α is finite.

If G is FC-nilpotent, M is nilpotent and A and B are hypercentral, then it was shown in [3] that G is also hypercentral. It was asked whether this result extends to the case when G is merely FC-hypercentral and M is hypercentral. The following corollary gives a positive answer to this question.

Corollary 1.2. *Let the FC-hypercentral group $G = AB = AM = BM$ be the product of three hypercentral subgroups A, B and M, where M is normal in G. Then G is hypercentral.*

Using standard arguments it is easy to deduce the following corollaries from Theorem 1.1 (see for example [2], proofs of Corollaries 6.3.9 and 6.3.11).

Corollary 1.3. *Let the group $G = AB$ be the product of two locally nilpotent subgroups A and B. If G has an ascending series with minimax factors, then each term of the Hirsch-Plotkin series of G is factorized. In particular, the Hirsch-Plotkin radical R of G is factorized, i.e. $R = (A \cap R)(B \cap R)$ and $A \cap B \subseteq R$.*

Corollary 1.4. *Let the group $G = AB$ be the product of two locally nilpotent subgroups A and B. If G has an ascending series with minimax factors, the factorizer $X(N) = AN \cap BN$ of every normal subgroup N of G is ascendant in G. In particular, the intersection $A \cap B$ is ascendant in G.*

The notation is standard and can for instance be found in [6] and [2]. In particular, the factorizer $X(N)$ of the normal subgroup N of the factorized group $G = AB$ is the subgroup $X(N) = AN \cap BN$; it is easy to see that $X(N) = A_1B_1 = A_1N = B_1N$ where $A_1 = A \cap BN$ and $B_1 = B \cap AN$.

2. Some lemmas

We will need several lemmas, some of which are special cases of our theorem.

Lemma 2.1. *Let G be a group and A a torsion-free abelian minimax normal subgroup of G. Suppose that the factor group G/A is locally nilpotent and for every non-trivial normal subgroup N of G the factor group $A/(A \cap N)$ is periodic. Then either G is locally nilpotent or G has finite Prüfer rank.*

Proof. It follows from [6], Theorem 10.35, and the corollary to Lemma 10.37, that the factor group $G/C_G(A)$ is minimax. Let $\hat{A}$ be the radicable hull of A. Then $\hat{A}$ has finite Prüfer rank and the action of G on A induces an action of G on $\hat{A}$ and we can construct the product $\hat{G} = \hat{A}G$ in which $\hat{A}$ is a normal subgroup and $G \cap \hat{A} = A$. Clearly $C_G(\hat{A}) = C_C(A)$.

Since G/A is locally nilpotent, the factor group $G/C_G(A)$ is hypercentral. If $C_G(A) = G$, then G is locally nilpotent and the lemma is proved. Let $C_G(A) \neq G$ and $gC_G(A)$ be a non-trivial central element of $G/C_G(A)$. Clearly the centralizer $C_A(g)$ is a normal subgroup of G, and the factor group $A/C_A(g)$ is periodic if $C_A(g) \neq 1$. Since $A/C_A(g)$ is isomorphic with $[A, g]$ and A is torsion-free, this implies $C_A(g) = 1$ and hence also $C_{\hat{A}}(g) = 1$. Thus $[\hat{A}, g]$ is isomorphic with $\hat{A}$ so that $\hat{A} = [\hat{A}, g]$ (see [5], Vol. 2, p. 153). In particular $\hat{A} = [\hat{A}, G]$. By Robinson [7], Theorem 4.5, the group $\hat{G}$ splits over $\hat{A}$ so that there exists a subgroup H of $\hat{G}$ such that $\hat{G} = H \ltimes \hat{A}$ is a semi-direct product of a subgroup H and the normal subgroup $\hat{A}$. Obviously $C_H(\hat{A})$ is a normal subgroup of $\hat{G}$ and by the hypothesis of the lemma $C_H(\hat{A}) \cap G = 1$. Since the factor group $H/C_H(\hat{A})$

is isomorphic with $\hat{G}/C_{\hat{G}}(\hat{A})$ and so also with $G/C_G(A)$, it is minimax. Therefore the factor group $\hat{G}/C_H(\hat{A})$ and thus also G have finite Prüfer rank. The lemma is proved.

Lemma 2.2. *Let G be an extension of a torsion-free abelian minimax group by a locally nilpotent group and suppose that $G = MA = MB = AB$ with three locally nilpotent subgroups A, B and M, where M is normal in G. Then the group G is locally nilpotent.*

Proof. Let X be an abelian minimax normal subgroup of G with minimal finite Prüfer rank such that the factor group G/X is locally nilpotent. If N is a normal subgroup of G which is maximal with the conditions that $N \cap X \subset X$ and $X/(N \cap X)$ is torsion-free, then the factor group $\bar{G} = G/N$ satisfies the hypothesis of Lemma 2.1. It follows that $\bar{G}$ has finite Prüfer rank. Moreover, $\bar{G}$ has a triple factorization $\bar{G} = \bar{M}\bar{A} = \bar{M}\bar{B} = \bar{A}\bar{B}$ with three locally nilpotent subgroups $\bar{A}$, $\bar{B}$ and $\bar{M}$. Hence $\bar{G}$ is locally nilpotent by [2], Theorem 6.3.8. But then the factor group $G/(N \cap X)$ is also locally nilpotent, since it is embedded into the direct product $(G/N) \times (G/X)$ of two locally nilpotent subgroups G/N and G/X. As $X/(N \cap X)$ is torsion-free and non-trivial, the Prüfer rank of $N \cap X$ is less than the Prüfer rank of X. This contradicts the choice of X. The lemma is proved.

Lemma 2.3. *Let the group G be an extension of a finite abelian group by an locally nilpotent group and suppose that $G = MA = MB = AB$ with three locally nilpotent subgroups A, B and M. Then G is locally nilpotent.*

Proof. Let X be a finite abelian normal subgroup of G with minimal order such that the factor group G/X is locally nilpotent. Then $[X, G] = X$ and by a theorem of Robinson (see [7], Corollary 3.5) the group G splits over X. Therefore there exists a subgroup H of G such that $G = H \ltimes X$. Obviously $C_H(X)$ is a normal subgroup of G and the factor group $\bar{G} = G/C_H(X)$ is finite. Since $\bar{G} = \bar{M}\bar{A} = \bar{M}\bar{B} = \bar{A}\bar{B}$ is a trifactorized group with nilpotent subgroups$\bar{A}$, $\bar{B}$ and $\bar{M}$, the group $\bar{G}$ is nilpotent by a result of Kegel (see [2], Corollary 2.5.11). Hence G is locally nilpotent, because it is embedded in the direct product $(G/X) \times (G/C_H(X))$. The lemma is proved.

The proof of the following lemma can be found in [4], Lemma 7.

Lemma 2.4. *Let $G = AM$ be the product of a subgroup A and a locally nilpotent normal subgroup M, and let M_1 and M_2 be subgroups of M such that $\langle M_1, A\rangle$ and $\langle M_2, A\rangle$ are locally nilpotent. Then also $\langle M_1, M_2, A\rangle$ is locally nilpotent.*

Lemma 2.5. *Let the group $G = MA = MB = AB$ be the product of three locally nilpotent subgroups A, B and M, where M is normal in G. If M has an ascending G-invariant series with minimax factors and the factors $A/(A \cap M)$ and $B/(B \cap M)$ are finitely generated, then G is locally nilpotent.*

Proof. By hypothesis the subgroup M has an ascending G-invariant series $1 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_\gamma = M$ whose factors are abelian and either finite or torsion-free minimax groups. There exists a least ordinal α such that G/M_α is locally nilpotent. If $\alpha = 0$, then G is locally nilpotent and the lemma is proved. Therefore let $\alpha > 0$. If α is not a limit ordinal, $G/M_{\alpha-1}$ is an extension of an abelian minimax group which is either finite or torsion-free by a locally nilpotent group. By Lemma 2.2 and Lemma 2.3 the factor group $G/M_{\alpha-1}$ is locally nilpotent, a contradiction. Hence α must be a limit ordinal and $M_\alpha = \bigcup M_\beta$ where $\beta < \alpha$.

By hypothesis there exist elements $a_1, \ldots, a_s$ of A and $b_1, \ldots, b_t$ of B such that $A = (A \cap M)\langle a_1, \ldots, a_s\rangle$ and $B = (B \cap M)\langle b_1, \ldots, b_t\rangle$. Moreover, there exist elements $a'_1, \ldots, a'_t$ of A, $b'_1, \ldots, b'_s$ of B and $m_1, \ldots, m_s$, $m'_1, \ldots, m'_t$ of M such that $b'_i = a_i m_i$ and $b_j = a'_j m'_j$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. Put $A^* = \langle a_1, \ldots, a_s, a'_1, \ldots, a'_t\rangle$, $B^* = \langle b_1, \ldots, b_t, b'_1, \ldots, b'_s\rangle$ and $M^* = \langle m_1, \ldots, m_s, m'_1, \ldots, m'_t\rangle$. Then we have

$$\langle A^*, B^*\rangle = \langle A^*, M^*\rangle = \langle B^*, M^*\rangle.$$

Since G/M_α is locally nilpotent, the group $\langle A^*, M^*\rangle/\langle A^*, M^*\rangle \cap M_\alpha$ is nilpotent and so the intersection $\langle A^*, M^*\rangle \cap M_\alpha$ is finitely generated as an $\langle A^*, M^*\rangle$-operator group. Hence $\langle A^*, M^*\rangle \cap M_\alpha = \langle A^*, M^*\rangle \cap M_\beta$ for some ordinal $\beta < \alpha$. Then the factor group $\langle A^*, M^*\rangle M_\beta/M_\beta$ is nilpotent. Passing to the factor group G/M_β we may even suppose that $M_\beta = 1$. Then the subgroup $\langle A^*, M^*\rangle$ is nilpotent and hence the subgroup $\langle A \cap M, A^*, M^*\rangle$ is locally nilpotent by Lemma 2.4. Similarly also $\langle B \cap M, A^*, B^*\rangle$ is locally nilpotent. Since $\langle B \cap M, B^*, M^*\rangle = \langle B \cap M, A^*, M^*\rangle$, another application of Lemma 2.4, yields that the subgroup $\langle A \cap M, B \cap M, \langle A^*, M^*\rangle\rangle$ is locally nilpotent. Now we have $A = (A \cap M)A^*$, $B = (B \cap M)B^*$ and

$$\langle A \cap M, B \cap M, \langle A^*, M^*\rangle\rangle = \langle A \cap M, B \cap M, \langle A^*, B^*\rangle\rangle = G.$$

Thus G is locally nilpotent. This contradiction proves the lemma.

Lemma 2.6. *Let the group $G = MA = MB = AB$ be the product of three locally nilpotent subgroups A, B and M, where M is normal in G and has an ascending G-invariant series with minimax factors. If M is non-trivial, then there exists a non-trivial normal subgroup K of G which is contained in M such that its factorizer $X = X(K)$ in G is locally nilpotent. If G is locally nilpotent, then K can be chosen as a cyclic central subgroup of G.*

Proof. It follows from the hypothesis that the subgroup M is hypercentral and has a non-trivial center Z if $M \neq 1$. Moreover, there exists a subgroup K of Z such that K is either a finite minimal normal subgroup of G or a torsion-free minimax group on which G acts rationally irreducibly. The factorizer $X(K)$ in G is locally nilpotent by [2], Theorem 6.3.7. In particular, if G is locally nilpotent, then K is a cyclic central subgroup of G.

3. Proof of the theorem

Let L be a normal subgroup of G contained in M such that the factorizer F of L in G is locally nilpotent, and L is maximal with these properties. If $L = M$ we are done. Assume that $L \subset M$ and consider the factor group $\bar{G} = G/L$. Then $\bar{G} = \bar{A}\bar{B} = \bar{A}\bar{M} = \bar{B}\bar{M}$, where the images modulo L are indicated by bars. By Lemma 2.6 there exists a subgroup K of M which is normal in G and properly contains L such that the factorizer $\bar{X} = X(\bar{K})$ in $\bar{G}$ is locally nilpotent. Let X denote the full preimage of $\bar{X}$ in G. Then the subgroup X is the factorizer of K in G which satisfies the hypotheses of the theorem and is not locally nilpotent. Thus, without loss of generality we may assume that $G = X$. Then the factor group $\bar{G} = \bar{A}\bar{B} = \bar{A}\bar{K} = \bar{B}\bar{K}$ is locally nilpotent. Repeating the above arguments we may assume by Lemma 2.6 that $\bar{K}$ is a cyclic central subgroup of $\bar{G}$. Obviously in this case the intersection $\bar{A} \cap \bar{B}$ is a normal subgroup of $\bar{G}$. Therefore its full preimage $F = AL \cap BL$ is a locally nilpotent normal subgroup of G and so is contained in the Hirsch-Plotkin radical R of G. In particular the intersection $A \cap B$ lies in R. Clearly $G = AB = AR = BR$. It is easy to see that the factor groups $\bar{A}/(\bar{A} \cap \bar{B})$ and $\bar{B}/(\bar{A} \cap \bar{B})$ are cyclic. Therefore the factor groups $A/(A \cap R)$ and $B/(B \cap R)$ are also cyclic and hence the group G is locally nilpotent by Lemma 2.5. This contradiction proves the theorem.

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BERNHARD AMBERG
FACHBEREICH MATHEMATIK
DER UNIVERSITÄT MAINZ
D–55099 MAINZ
GERMANY

YAROSLAV P. SYSAK
INSTITUTE OF MATHEMATICS
UKRAINIAN ACADEMY OF SCIENCES
252601 KIEV
UKRAINE

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