On derivatives of an algebraic polynomial of best approximation with weight

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Let us denote by $L_p(-\infty, \infty)$ — as usual — the Banach space of measurable functions in $(-\infty, \infty)$ with norm

$$\|f(x)\|_p = \Big\{ \int_{-\infty}^{\infty} |f(x)|^p\, dx \Big\}^{1/p} \quad (1 \leqq p < \infty)$$

resp. for $p=\infty$,

$$\|f(x)\|_\infty = \operatorname*{vrai\cdot sup}_{-\infty < x < \infty} |f(x)|.$$

Let P_n be a set of algebraic polynomials of degree not greater than n $(n=0, 1, \ldots)$. Let

$$v_{\alpha,\beta}(x) = (1+|x|^\alpha)^{\beta/2\alpha} e^{-|x|\alpha/2}, \tag{1}$$
$$-\infty < x < \infty \quad \alpha \geqq 2, \quad \beta \geqq 0.$$

For an f satisfying $v_{\alpha,\beta} f \in L_p(-\infty, \infty)$, we define

$$E_n = E_n^{(p)}(v_{\alpha,\beta}; f) = \inf_{q \in P_n} \|v_{\alpha,\beta}(f-g)\|_p \quad n = 0, 1, \ldots. \tag{2}$$

We have for each $v_{\alpha,\beta} f \in L_p(-\infty, \infty)$,

$$E_n^{(p)}(v_{\alpha,\beta}; f) \to 0 \quad (n \to \infty). \tag{3}$$

It is known that there exists a unique

$$q_n(x) = q_n(f; x) = q_n(L_p; f; x) \quad \text{that} \quad E_n^{(p)}(v_{\alpha,\beta}; f) = \|v_{\alpha,\beta}(f-q_n)\|_p. \tag{4}$$

The polynomial $g_n(L_p; f; x)$ is called the n-th polynomial of best approximation of $f(x)$ with the weight $v_{\alpha,\beta}(x)$ in L_p-space.

The characterization of

$$E_n^{(p)}(v_{\alpha,\beta}; f) = O(n^{-\gamma}) \tag{5}$$

by the integral modulus of continuity of f is due to professor GÉZA FREUD [1]. In this paper of us, we give an alternative condition for (5) by the derivates of the polynomials of best approximation. The analogue of our Theorem for trigonometric approximation of periodic function is due to M. ZAMANSKY [6], G. I. SUNOUCHI [5] and for algebraic approximation to S. PAWELKE [4].

Theorem. *Let $v_{\alpha,\beta} f \in L_p(-\infty, \infty)$. Let r be a positive integer, and let γ be such positive real number, that $(1-1/\alpha)r-\gamma>0$. For the polynomial $g_n(f;x)= =g_n(L_p; f; x)$ the following statements are equivalent:*

(6) (i) $E_n^{(p)}(v_{\alpha,\beta}; f) = O(n^{-\gamma})$.

(7) (ii) $\|v_{\alpha,\beta} q_n^{(r)}(f)\|_p = O(n^{(1-1/\alpha)r-\gamma})$.

PROOF. The proof will be based on the following inequalities:

1) For each real λ there is a constant *) M_λ with the property that for each sequence $0<u_k \leqq u_{k+1} \leqq \ldots \leqq u_l$, such that $2 \leqq u_i/u_{i-1} \leqq 4$ for $k<i \leqq l$, and for each positive decreasing function $\varphi(u)$ defined for $u \geqq 0$,

$$\sum_{i=k}^{l} u_i^\lambda \varphi(u_i) \leqq M_\lambda \sum_{n=[(1/2)u_k]}^{u_l-1} (n+1)^{\lambda-1} \varphi(n). \tag{8}$$

(See G.G. LORENTZ [4, p. 58—59])

2) By a not yet published result of Freud **) for each $p_n \in P_n$, we have

$$\|v_{\alpha,\beta} p_n^{(r)}\|_p \leqq M^{(1)}_{p,r,\alpha,\beta} n^{(1-1/\alpha)r} \|v_{\alpha,\beta} p_n\|_p. \tag{9}$$

3) If f is an r-time iterated integral function of $f^{(r)}$, and $v_{\alpha,\beta} f^{(r)} \in L_p(-\infty, \infty)$ then

$$E_n^{(p)}(v_{\alpha,\beta}; f) \leqq M^{(2)}_{p,r,\alpha,\beta} n^{(1/\alpha-1)r} \|v_{\alpha,\beta} f^{(r)}\|_p. \tag{10}$$

(See G. FREUD [1]).

a) (i)$\to$(ii): For $n \geqq 1$, let k be such integer, that $2^k \leqq n < 2^{k+1}$. We have

$$\begin{aligned} q_n^{(r)}(x) &= [q_n(x)-q_{2^k}(x)]^{(r)} + [q_{2^k}(x)-q_{2^{k-1}}(x)]^{(r)} + \ldots + \\ &\quad + [q_2(x)-q_1(x)]^{(r)} + [q_1(x)-q_0(x)]^{(r)}. \end{aligned} \tag{11}$$

For $E_n = E_n^{(p)}(v_{\alpha,\beta}; f)$, we notice that

$$\|v_{\alpha,\beta}(q_{2^i}-q_{2^{i-1}})\|_p \leqq 2E_{2^{i-1}} \quad (i=1, 2, \ldots, k), \quad \|v_{\alpha,\beta}(q_n-q_{2^k})\|_p \leqq 2E_{2^k} \tag{12}$$

Now, from (11), (9) and (12), using the inequality (8) we get

$$\begin{aligned} \|v_{\alpha,\beta} q_n^{(r)}\|_p &\leqq M^{(1)}_{p,r,\alpha,\beta} \{n^{(1-1/\alpha)r} 2E_{2^k} + 2^{k(1-1/\alpha)r} 2E_{2^{k-1}} + \ldots + 2^{(1-1/\alpha)r} 2E_1 + 2E_0\} \leqq \\ &\leqq 2^{(1-1/\alpha)r+1} M^{(1)}_{p,r,\alpha,\beta} \left[\sum_{i=0}^{k} 2^{i(1-1/\alpha)r} E_{2^i} + E_0\right] \leqq \\ &\leqq 2^{(1-1/\alpha)r+2} M^{(1)}_{p,\alpha,\beta,r} M_{r,\alpha} \sum_{l=0}^{2^k-1} (l+1)^{(1-1/\alpha)r-1} E_l \leqq \\ &\leqq 2^{(1-1/\alpha)r+2} M^{(1)}_{p,r,\alpha,\beta} M_{r,\alpha} \sum_{l=0}^{n-1} (l+1)^{(1-1/\alpha)r-1} E_l. \end{aligned} \tag{13}$$

*) The $M^{(k)}_{\lambda,p,\ldots}$ denote positive constants, which depend only on $\lambda, p, \ldots$

**) Special cases are delt in [2]. The more general case was included in prof. Freud's lectures at the Szeged University in the year 1973/74.

Hence follows that, if $E_l=O(l^{-\gamma})$, then since $(1-1/\alpha)r-1-\gamma>-1$, we have

$$\|v_{\alpha,\beta}q_n^{(r)}\|_p = O(n^{(1-1/\alpha)r-\gamma}),$$

which was to be proved.

b) (ii) $\to$ (i): If $\|v_{\alpha,\beta}q_n^{(r)}(f)\|_p \leqq An^{(1-1/\alpha)r-\gamma}$,

then from (10) we obtain

$$\|v_{\alpha,\beta}\{q_{2n}(f)-q_n[q_{2n}(f)]\}\|_p \leqq M^{(2)}_{p,r,\alpha,\beta}\cdot A\cdot n^{-\gamma}. \tag{14}$$

Furthermore, since $g_n[g_{2n}(f)]\in P_n$, we have

$$\begin{aligned}\|v_{\alpha,\beta}\{q_{2n}(f)-q_n[q_{2n}(f)]\}\|p \geqq \|v_{\alpha,\beta}\{f-q_n[q_{2n}(f)]\}\|_p-\\ -\|v_{\alpha,\beta}[f-q_{2n}(f)]\|_p \geqq E_n^{(p)}(v_{\alpha,\beta};f)-E_{2n}^{(p)}(v_{\alpha,\beta};f).\end{aligned} \tag{15}$$

From (3). (14) and (15) we get

$$E_{2^k}=\sum_{i=k}^{\infty}\{E_{2^i}-E_{2^{i+1}}\}\leqq M^{(2)}_{p,r,\alpha,\beta}\cdot A\cdot\sum_{i=k}^{\infty}2^{-i\gamma}=O(2^{-k\gamma}),$$

from which follows, that $E_n=O(n^{-\gamma})$. This completes our proof.

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(Received July 20, 1974.)