On the functional equation $f(x+y)+g(xy) = h(x)+h(y)$

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1. *Introduction*

The following problem is due to Professor Z. DARÓCZY (see [1]). Let $\mathbf{R}$ denote the set of real numbers and let $\mathbf{R}_+ = \{x | x \in \mathbf{R},\ x>0\}$. Find all continuous functions $f, g, h: \mathbf{R}_+ \to \mathbf{R}$ satisfying $f(x+y)+g(xy)=h(x)+h(y)$ for all $x, y \in \mathbf{R}_+$. This equation has been discussed under other assumptions too (see [2], [3]), but its general solution on $\mathbf{R}_+$ has not been found. The purpose of this paper is to give the general solution of the above mentioned equation on $\mathbf{R}_+$.

First, we shall solve the following problem:

A. Find all functions $f: \mathbf{R}_+ \to \mathbf{R}$ satisfying

$$f(x+\alpha y)-f(x+\alpha)-f(\alpha y+1) = f(y+\alpha x)-f(y+\alpha)-f(\alpha x+1) \tag{1}$$

for all $x, y, \alpha \in \mathbf{R}_+$.

With the help of the solution of Problem A. we can easily give the solution of our original problem which can be formulated as follows:

B. Find all functions $f, g, h: \mathbf{R}_+ \to \mathbf{R}$ satisfying

$$f(x+y)+g(xy) = h(x)+h(y) \tag{2}$$

for all $x, y \in \mathbf{R}_+$.

2. *Problem* A.

The following theorem will be important in view of the solution of Problem A.

Theorem 1. *Let*

$$f: \mathbf{R}_+ \to \mathbf{R}, \quad H: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}, \quad T: \mathbf{R}_+ \to \mathbf{R}$$

satisfy the equations

$$f(x+y)-f(x) = H(x, y)+T(y) \quad (x, y \in \mathbf{R}_+) \tag{a}$$

$$H(x+y, z) = H(x, z)+H(y, z) \quad (x, y, z \in \mathbf{R}_+). \tag{b}$$

Then there exist

$$D: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}, \quad B: \mathbf{R}_+ \to \mathbf{R}, \quad C \in \mathbf{R}$$

such that

$$(c) \qquad D(x, y) = D(y, x) \quad (x, y \in \mathbf{R}_+)$$

$$(d) \qquad D(x+y, z) = D(x, z) + D(y, z) \quad (x, y, z \in \mathbf{R}_+)$$

$$(e) \qquad B(x+y) = B(x) + B(y) \quad (x, y \in \mathbf{R}_+)$$

$$(f) \qquad f(x) = D(x, x) + B(x) + C \quad (x \in \mathbf{R}_+).$$

PROOF. Let

$$F(x, y) = f(x+y) - f(x) - f(y) \quad (x, y \in \mathbf{R}_+).$$

Since F is an additive deviation (see [4]) and by (a) we have

$$F(x, y) = H(x, y) + T(y) - f(y) \quad (x, y \in \mathbf{R}_+)$$

therefore

$$H(x+y, z) + H(x, y) + T(y) - f(y) = H(x, y+z) + T(y+z) - f(y+z) + H(y, z).$$

According to (b) it follows that

$$(3) \quad H(x, y) + H(x, z) + T(y) - f(y) = H(x, y+z) + T(y+z) - f(y+z) \quad (x, y, z \in \mathbf{R}_+).$$

After interchanging y and z in (3) we can see that

$$T(y) - f(y) = T(z) - f(z) \quad (y, z \in \mathbf{R}_+)$$

and so there exists $C \in \mathbf{R}$ such that

$$(4) \qquad T(y) = f(y) - C \quad (y \in \mathbf{R}_+).$$

It implies that

$$(a') \qquad f(x+y) - f(x) - f(y) + C = H(x, y) \quad (x, y \in \mathbf{R}_+).$$

Let now

$$D(x, y) = \frac{1}{2} H(x, y) \quad \text{and} \quad B(x) = \frac{1}{2}[4f(x) - f(2x) - 3C] \quad (x, y \in \mathbf{R}_+).$$

Then — by (a') — we get (c), furthermore (b) implies (d). Using (a') and the properties of D we obtain (e) and (f). ∎

We remark that the converse of Theorem 1. is also true.

Investegating equation (1) we obtain the following result:

Theorem 2. *Let* $f: \mathbf{R}_+ \to \mathbf{R}$ *be a solution of* (1) *and let* $\alpha \in \mathbf{R}_+$ *be fixed, moreover let*

$$m_\alpha = \max\{-1-\alpha-\alpha^2, -2-\alpha\}, \quad D_\alpha = \{(u, v) | (u, v) \in \mathbf{R}^2, \quad u > -1, v > -\alpha\}$$

$$F_\alpha(u) = f(u+1+\alpha+\alpha^2) - f(u+2+\alpha) \quad (u \in]m_\alpha, \infty[).$$

Then

$$(5) \qquad F_\alpha(y+t) + F_\alpha(0) = F_\alpha(y) + F_\alpha(t)$$

holds for all $(y, t) \in D_\alpha$.

PROOF. Let

$$x\tilde{\alpha}y = f(x+\alpha y)-f(x+\alpha)-f(\alpha y+1) \quad (x, y\in \mathbf{R}_+). \tag{6}$$

A simple calculation shows that

$$x\tilde{\alpha}y = y\tilde{\alpha}x \quad (x, y\in \mathbf{R}_+) \tag{1'}$$

$$\left(\frac{t}{\alpha}+1+\alpha\right)\tilde{\alpha}(y+1)+\left(\frac{t}{\alpha}+1\right)\tilde{\alpha}2 = \left(\frac{t}{\alpha}+1\right)\tilde{\alpha}(y+2)+(1+\alpha)\tilde{\alpha}(y+1) \quad ((y, t)\in D_\alpha). \tag{7}$$

Hence,

$$(y+1)\tilde{\alpha}\left(\frac{t}{\alpha}+1+\alpha\right)+2\tilde{\alpha}\left(\frac{t}{\alpha}+1\right) = (y+2)\tilde{\alpha}\left(\frac{t}{\alpha}+1\right)+(y+1)\tilde{\alpha}(1+\alpha) \quad ((y, t)\in D_\alpha).$$

By (6) and the definition of F_α, this implies that (5) holds for all $(y, t)\in D_\alpha$. ∎

Now we use an extension theorem (see [5]) and we get the following result:

Theorem 3. *Let $f\colon \mathbf{R}_+\to\mathbf{R}$ be a solution of* (1) *and let $\beta\in\mathbf{R}_+$ be fixed. Then there exist functions $L_\beta, T\colon \mathbf{R}_+\to\mathbf{R}$ such that*

$$L_\beta(u+v) = L_\beta(u)+L_\beta(v) \quad (u, v\in\mathbf{R}_+) \tag{8}$$

and

$$f(u+\beta)-f(u) = L_\beta(u)+T(\beta) \quad (u\in\mathbf{R}_+).$$

PROOF. Let

$$L_\beta(t) = F_{\sqrt{\beta+1}}(t)-F_{\sqrt{\beta+1}}(0) \quad (t\in]m_{\sqrt{\beta+1}}, \infty[).$$

Then — applying Theorem 2. — from (5) we obtain (8). It is known by [5] that there exists one and only one function $\bar{L}_\beta\colon \mathbf{R}\to\mathbf{R}$ such that

$$\bar{L}_\beta(u+v) = \bar{L}_\beta(u)+\bar{L}_\beta(v) \quad (u, v\in\mathbf{R})$$

and

$$\bar{L}_\beta(u) = L_\beta(u) \quad (u\in]m_{\sqrt{\beta+1}}, \infty[).$$

Let

$$T(t) = \bar{L}_t\left(-2-\sqrt{t+1}\right)+F_{\sqrt{t+1}}(0) \quad (t\in\mathbf{R}_+).$$

If $u\in\mathbf{R}_+$ then $u-2-\sqrt{\beta+1}\in]m_{\sqrt{\beta+1}}, \infty[$, therefore

$$\begin{aligned} L_\beta(u)+T(\beta) &= \bar{L}_\beta(u)+\bar{L}_\beta\left(-2-\sqrt{\beta+1}\right)+F_{\sqrt{\beta+1}}(0) = \\ &= \bar{L}_\beta\left(u-2-\sqrt{\beta+1}\right)+F_{\sqrt{\beta+1}}(0) = \\ &= L_\beta\left(u-2-\sqrt{\beta+1}\right)+F_{\sqrt{\beta+1}}(0) = \\ &= F_{\sqrt{\beta+1}}\left(u-2-\sqrt{\beta+1}\right) = f(u+\beta)-f(u). \end{aligned}$$

∎

Theorem 3. allows to apply Theorem 1. and so we get the following result as the solution of Problem A.

Theorem 4. *Let* $f: \mathbf{R}_+ \to \mathbf{R}$ *be a solution of* (1). *Then there exist additive functions* $A, B: \mathbf{R}_+ \to \mathbf{R}$ *and* $C \in \mathbf{R}$ *such that*

$$f(x) = A(x^2) + B(x) + C \quad (x \in \mathbf{R}_+). \tag{9}$$

PROOF. Let T and L_β be as in Theorem 3. and let $H(x, y) = L_y(x)$ $(x, y) \in \mathbf{R}_+)$. Then f, T and H satisfy (a) and (b) in Theorem 1. Thus there exist

$$D: \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}, \quad B: \mathbf{R}_+ \to \mathbf{R}, \quad C \in \mathbf{R}$$

with the properties $(c)-(f)$. By (f) we obtain from (1)

$$D(x, \alpha y) - D(x, \alpha) - D(\alpha y, 1) = D(\alpha x, y) - D(\alpha, y) - D(\alpha x, 1) \quad (x, y, \alpha \in \mathbf{R}_+). \tag{10}$$

If we write $x+1$ instead of x in (10) and use the properties of D we obtain

$$D(x, \alpha y) - D(x, \alpha) = D(\alpha x, y) - D(\alpha x, 1) \quad (x, y, \alpha \in \mathbf{R}_+).$$

Combining this equation and (10) we get

$$D(\alpha y, 1) = D(\alpha, y) \quad (\alpha, y \in \mathbf{R}_+). \tag{11}$$

Let $A(t) = D(t, 1)$, $(t \in \mathbf{R}_+)$, then — by (11) — we obtain (9). ∎

We remark that the converse of Theorem 4. is also true.

3. *Problem B.*

The following theorem gives the solution of Problem B.

Theorem 5. Let $(f, g, h): \mathbf{R}_+ \to \mathbf{R}^3$ be a solution of (2). Then there exist additive functions $A, B: \mathbf{R}_+ \to \mathbf{R}$ and $\varphi: \mathbf{R}_+ \to \mathbf{R}$ with the property $\varphi(xy) = \varphi(x) + \varphi(y)$, $(x, y \in \mathbf{R}_+)$ and $C, C_1, C_2 \in \mathbf{R}$ such that

$$f(x) = A(x^2) + B(x) + C \quad (x \in \mathbf{R}_+) \tag{i}$$

$$g(x) = \varphi(x) - 2A(x) + C_1 \quad (x \in \mathbf{R}_+) \tag{ii}$$

$$h(x) = A(x^2) + B(x) + \varphi(x) + C_2 \quad (x \in \mathbf{R}_+). \tag{iii}$$

PROOF. Putting $y=1$ in (2), we obtain

$$h(x) = f(x+1) + g(x) - h(1) \quad (x \in \mathbf{R}_+). \tag{12}$$

Thus (2) can be written in the form

$$f(x+y) - f(x+1) - f(y+1) = g(x) + g(y) - g(xy) - 2h(1) \quad (x, y \in \mathbf{R}_+). \tag{13}$$

By [4], it is known that the right hand side of (13) is a multiplicative deviation and so we have (1). According to Theorem 4. (9) implies (i). Using (13) and (i) we get

$$2A(xy) - 2A(x) - 2A(y) - 2A(1) - 2B(1) + 2h(1) = g(x) + g(y) - g(xy) + C. \tag{14}$$

Let $C_1=2[h(1)-A(1)-B(1)]-C$ and

(15) $$\varphi(x) = g(x)+2A(x)-C_1 \quad (x\in\mathbf{R}_+).$$

Then (14) implies that $\varphi(xy)=\varphi(x)+\varphi(y)$ holds for all $x, y\in\mathbf{R}_+$ and by (15) we obtain (ii). Finally, let $C_2=\dfrac{C+C_1}{2}$. Then (12) implies (iii). ∎

We remark that if $C+C_1=2C_2$ then the converse of Theorem 5. is also true.

4. *Remark*

The equations

$$A(x) = \frac{1}{2}[h(x)-f(x)-g(x)]+\frac{C+C_1-C_2}{2} \quad (x\in\mathbf{R}_+)$$

$$\varphi(x) = h(x)-f(x)+C-C_2 \quad (x\in\mathbf{R}_+)$$

derived from (i), (ii) and (iii) show that if the functions f, g and h have some analytic property on a certain subset of $\mathbf{R}_+$ then — usually — the functions A and φ also have the same property. For example, if the functions f, g and h are continuous at the point $x, (x\in\mathbf{R}_+)$ or measurable on a certain measurable subset E of positive measure of $\mathbf{R}_+$ then both functions A and φ are continuous at x or measurable on E, respectively. Thus there exist $\alpha, \gamma\in\mathbf{R}$ such that $A(x)=\alpha x, \varphi(x)=\gamma \ln x$ for all $x\in\mathbf{R}_+$, moreover by (i) there exists $\beta\in\mathbf{R}$ such that $B(x)=\beta x$ for all $x\in\mathbf{R}_+$. In these cases, of course

$$f(x) = \alpha x^2+\beta x+C$$

$$g(x) = \gamma \ln x-2\alpha x+C_1$$

$$h(x) = \alpha x^2+\beta x+\gamma \ln x+C_2$$

for all $x\in\mathbf{R}_+$.

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