Convergence of Hermite—Fejér interpolation polynomial on the extended nodes

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1. In 1969 BERMAN [1] proved a surprising result that the *Hermite—Fejér interpolation process* $H_n(f, x)$ of degree $\leqq 2n-1$ defined by the conditions:

$$H_n(f, x_k) = f(x_k), \quad H_n'(f, x_k) = 0, \quad k = 1, 2, \dots, n \tag{1}$$

constructed on the nodes

$$x_0 = 1, \quad x_k = \cos\frac{2k-1}{2n}\pi, \quad k = 1, 2, 3, \dots, n \tag{2}$$

or

$$x_{n+1} = -1, \quad x_k = \cos\frac{2k-1}{2n}\pi, \quad k = 1, 2, \dots, n \tag{3}$$

diverges for a simple function $f(x)=|x|$ at $x=0$, while according to the well known result of FEJÉR [3] the polynomial $H_n(f, x)$ constructed on the nodes $x_k = \cos\frac{2k-1}{2n}\pi$, $k=1, 2, \dots, n$, the roots of the Tchbycheff polynomial of the first kind, $T_n(x)$, converges to a continuous function $f(x)$ in $[-1, 1]$. Recently we [6] proved the divergence of $H_n(f, x)$ for nodes (2) or (3) every where in $[-1, 1]$.

The point system (2) or (3) is the extension of the Tchebycheff nodes by $+1$ or -1 and the Tchebycheff nodes are best interpolation nodes. It appears due to the result of BERMAN [1] that the addition of $+1$ or -1 in any system of nodes disturbs the convergence of the Hermite—Fejér interpolation polynomial. In this paper we show that this is not always true. We prove

Theorem 1. *The Hermite—Fejér interpolation process* $H_n(f, x)$ *constructed on the points* $\cos\frac{2k-1}{2n+1}\pi$ $(k=0, \dots, n+1)$ *converges uniformly to* $f(x)$ *in* $[-1, 1]$, *when* $f(x)$ *is continuous in* $[-1, 1]$.

Theorem 2. *The Hermite—Fejér interpolation process* $H_n(f, x)$ *constructed on the points* $\cos\frac{2k}{2n+1}\pi$ $(k=0, \dots, n)$ *converges uniformly to* $f(x)$ *in* $[-1, 1]$, *when* $f(x)$ *is continuous in* $[-1, 1]$.

The proof of theorem 2 runs exactly on the same lines as proof of theorem 1. So we omit the proof.

2. For our purpose we consider the system of nodes

$$x_{n+1}=-1, \quad x_k=\cos\frac{2k-1}{2n+1}\pi, \quad k=1,2,3,\dots,n. \tag{4}$$

The points $\cos\frac{2k-1}{2n+1}\pi$ are the roots of Jacobi polynomial $P_n^{(-1/2,1/2)}(x)$, which is identical with

$$W_n(x)=\text{const}\left[\frac{\cos(2n+1)\theta/2}{\cos\theta/2}\right]_{x=\cos\theta},$$

satisfying the differential equation

$$(1-x^2)W_n''(x)+(1-2x)W_n'(x)+n(n+1)W_n(x)=0. \tag{5}$$

It is well known that

$$\frac{W_n'(x)}{W_n(x)}=\sum_{k=1}^{n}\frac{1}{x-x_k}$$

and

$$\frac{W_n'^2(x)-W_n''(x)W_n(x)}{W_n^2(x)}=\sum_{k=1}^{n}\frac{1}{(x-x_k)^2},$$

which give

$$\sum_{k=1}^{n}\frac{1}{(1+x_k)}=\frac{n(n+1)}{3} \tag{6}$$

$$\sum_{k=1}^{n}\frac{1}{(1-x_k)}=n(n+1) \tag{7}$$

$$\frac{2(1+x)W_n^2(x)}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)}=\frac{\sin(2n+1)\theta}{(2n+1)\sin\theta}-\frac{\cos^2(2n+1)\theta/2}{(2n+1)^2\cos^2\theta/2} \tag{8}$$

and

$$\begin{aligned}\frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}&=1-\frac{(2\sin^2\theta/2-\cos^3\theta/2)\sin(2n+1)\theta}{(2n+1)\sin\theta}+\\&+\frac{(2\cos 3\theta/2-\cos\theta)\cos^2(2n+1)\theta/2}{(2n+1)^2\cos^2\theta/2}.\end{aligned} \tag{9}$$

3. Hermite—Fejér interpolation process $H_n(f,x)$ constructed on the nodes (4) is given by

$$\begin{aligned}H_n(f,x)&=[1+2/3n(n+1)(x+1)]\frac{W_n^2(x)}{W_n^2(-1)}f(-1)+\\&+2\sum_{k=1}^{n}f(x_k)\left[1-\frac{1}{1-x_k^2}(x-x_k)\right]\frac{(1+x)^2W_n^2(x)(1-x_k)}{(2n+1)^2(x-x_k)^2}.\end{aligned} \tag{10}$$

To prove theorem 1, we need the following lemmas.

Lemma 1. *For any* $f \in C$ *and* $-1 < x < 1$

$$\lim_{n\to\infty} \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{x-x_k} = 0, \tag{11}$$

where

$$x_k = \cos\frac{2k-1}{2n+1}\pi, \quad k = 1, 2, 3, \dots, n.$$

Proof. First, we prove this lemma, when $f(x)$ is a polynomial. So it suffices to take $f(x) = x^j$, j finite integer $\geqq 0$. For $j=0$ (11) follows at once from (8). For $j \geqq 1$, we set

$$A_{jn} = \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{x_k^j}{(x-x_k)} = \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{\{x-(x-x_k)\}^j}{(x-x_k)} =$$

$$= \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \left\{ \sum_{k=1}^{n} \frac{x^j}{(x-x_k)} + \sum_{k=1}^{n} \sum_{i=1}^{j} {}_jC_i (-1)^i (x-x_k)^i x^{j-i} \right\}.$$

Since

$$\left| \sum_{k=1}^{n} \sum_{i=1}^{j} {}_jC_i (-1)^i (x-x_k)^i x^{j-i} \right| \leqq 2^j n (2^j - 1),$$

therefore,

$$\left| A_{jn} - \frac{2(1+x)W_n^2(x)x^j}{(2n+1)^2} \sum_{k=1}^{n} \frac{1}{x-x_k} \right| \leqq \frac{2(1+x)W_n^2(x)2^j n(2^j-1)}{(2n+1)^2}.$$

Hence

$$\lim_{n\to\infty} A_{jn} = 0,$$

owing to (8) and the inequality $(1+x)W_n^2(x) \leqq 2$.

Now, let $f(x) \in C$. We know that for $\varepsilon > 0$, there exists a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon, \quad \text{for} \quad -1 \leqq x \leqq 1.$$

Setting

$$R_n(f) = \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(x-x_k)},$$

we get

$$|R_n(f)| \leqq |R_n(f-P)| + |R_n(P)|. \tag{12}$$

Since (11) holds, when $f(x)$ is a polynomial, therefore there exists a number N such that

$$|R_n(P)| < \varepsilon \quad \text{for all} \quad n \geqq N. \tag{13}$$

Further

$$(14)\qquad |R_n(f-P)| \leqq \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{|f(x_k)-P(x_k)|}{|x-x_k|} \leqq$$

$$\leqq \frac{2\varepsilon(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{|x-x_k|}{(x-x_k)^2} \leqq \frac{4\varepsilon(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} =$$

$$= 4\varepsilon\left\{\frac{1}{\sin^2\theta} - \frac{(2\sin^2\theta/2-\cos^3\theta/2)\sin(2n+1)\theta}{(2n+1)\sin^3\theta} + \frac{2(\cos^3\theta/2-\cos\theta)\cos^2(2n+1)\theta/2}{(2n+1)^2\sin^2\theta\cos^2\theta/2}\right\},$$

$$-1 < x < 1.$$

Hence from (13), (14) and (12) it follows that

$$\lim_{n\to\infty} R_n(f) = 0.$$

Thus lemma 1 is proved.

Lemma 2. *For any* $f\in C$ *and* $-1<x<1$

$$(15)\qquad \lim_{n\to\infty} \frac{2(1+x)(1-x^2)W_n(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(x-x_k)^2} = f(x),$$

where

$$x_k = \cos\frac{2k-1}{2n+1}\pi, \quad k = 1, 2, \ldots, n.$$

Proof. As in the proof of lemma 1, it is sufficient to prove (15) only for $f(x)=x^j$, $j=0, 1, 2, \ldots$. For $f(x)=1$, (15) follows atonce from (9). Further let $f(x)=x$, then

$$B_{1n} = \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{x_k}{(x-x_k)^2} =$$

$$= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \left\{\sum_{k=1}^{n} \frac{x}{(x-x_k)^2} - \sum_{k=1}^{n} \frac{1}{x-x_k}\right\}.$$

Hence

$$\lim_{n\to\infty} B_{1n} = x$$

owing to (8) and (9).

Now, let $f(x)=x^j$, $j\geqq 2$. Then

$$B_{jn} = \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{x_k^j}{(x-x_k)^2} =$$

$$= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{\{x-(x-x_k)\}^j}{(x-x_k)^2} =$$

$$= \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \left\{\sum_{k=1}^{n} \frac{x^j}{(x-x_k)^2} - \sum_{k=1}^{n} \frac{jx^{j-1}}{(x-x_k)} + \right.$$

$$\left. + \sum_{k=1}^{n} \sum_{i=2}^{j} jC_i(-1)^i x^{j-i}(x-x_k)^{i-2}\right\},$$

which gives

$$\left|B_{jn}-\frac{2(1+x)(1-x^2)W_n^2(x)x^j}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}+\right.$$

$$\left.+\frac{2(1+x)(1-x^2)W_n^2(x)jx^{j-1}}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{x-x_k}\right| \leqq$$

$$\leqq \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}\left|\sum_{k=1}^{n}\sum_{i=2}^{j} jC_i(-1)^i x^{j-i}(x-x_k)^{i-2}\right|.$$

But

$$\left|\sum_{k=1}^{n}\sum_{i=2}^{j} jC_i(-1)^i x^{j-i}(x-x_k)^{i-2}\right| < 2^{j-2}n(2^j-1-j),$$

therefore,

$$\left|B_{jn}-\frac{2x^j(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}+\right.$$

$$\left.+\frac{2jx^{j-1}(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}\sum_{k=1}^{n}\frac{1}{x-x_k}\right| < \frac{2^{j-1}n(2^j-1-j)(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2}.$$

Taking limit, we get

$$\lim_{n\to\infty} B_{jn} = x^j$$

owing to (8), (9) and the inequality $(1+x)W_n^2(x)\leqq 2$.

Thus we have proved that (15) is true when $f(x)$ is a polynomial. Further by repeating the arguments as in the proof of lemma 1, we can prove that (15) is true for arbitrary $f(x)\in C[-1, 1]$.

Similarly we can prove

Lemma 3. *For any* $f\in C$ *and* $x_k=\cos\frac{2k-1}{2n+1}\pi,\ k=1, 2, \ldots, n$

$$\lim_{n\to\infty}\frac{1}{(2n+1)^2}\sum_{k=1}^{n}\frac{f(x_k)}{1+x_k}=\frac{f(-1)}{12} \tag{16}$$

and

$$\lim_{n\to\infty}\frac{1}{(2n+1)^2}\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k}=\frac{f(1)}{4}. \tag{17}$$

Proof *of theorem* 1. From (10), we have

$$H_n(f, x) = \left[1+\frac{2}{3}n(n+1)(x+1)\right]\frac{W_n^2(x)}{W_n^2(-1)}f(-1)+$$
$$+\frac{2(1+x)^2W_n^2(x)}{(2n+1)^2}\sum_{k=1}^{n}f(x_k)\left\{\frac{1-x_k}{(x-x_k)^2}-\frac{1}{(x-x_k)(1+x_k)}\right\}. \tag{18}$$

Since

$$(1+x)\left[\frac{1-x_k}{(x-x_k)^2}-\frac{1}{(x-x_k)(1+x_k)}\right]=\frac{(1-x^2)}{(x-x_k)^2}+\frac{x}{x-x_k}-\frac{1}{1+x_k},$$

therefore,

$$H_n(f, x) = \left[1 + \frac{2}{3} n(n+1)(x+1)\right] \frac{W_n^2(x)}{(2n+1)^2} f(-1) - \tag{19}$$

$$\frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1+x_k} + \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \times \left\{(1-x^2) \sum_{k=1}^{n} \frac{f(x_k)}{(x-x_k)^2} + x \sum_{k=1}^{n} \frac{f(x_k)}{x-x_k}\right\}.$$

Now taking limit, we get

$$\lim_{n\to\infty} H_n(f, x) = \lim_{n\to\infty} \left[1 + \frac{2}{3} n(n+1)(x+1)\right] \frac{W_n^2(x)}{(2n+1)^2} f(-1) - \tag{20}$$

$$- \lim_{n\to\infty} \frac{2(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1+x_k} + \lim_{n\to\infty} \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(x-x_k)^2} +$$

$$+ \lim_{n\to\infty} \frac{2x(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{x-x_k} =$$

$$= \lim_{n\to\infty} (1+x)W_n^2(x) \left[\frac{f(-1)}{6} - \lim_{n\to\infty} \frac{2}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1+x_k}\right] +$$

$$+ \lim_{n\to\infty} \frac{2(1+x)(1-x^2)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(x-x_k)^2} + 2x \lim_{n\to\infty} \frac{(1+x)W_n^2(x)}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{x-x_k}.$$

Applying lemmas $1-3$ to (20), we get

$$\lim_{n\to\infty} H_n(f, x) = f(x) \quad \text{for} \quad -1 < x < 1$$

owing to the inequality $(1+x)\, W_n^2(x) \leqq 2$. Again putting $x=1$ in (10), we have

$$H_n(f, 1) = \left[1 + \frac{4}{3} n(n+1)\right] \frac{f(-1)}{(2n+1)^2} + \frac{4}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1-x_k} - \frac{4}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1+x_k}. \tag{21}$$

Taking limit, we get

$$\lim_{n\to\infty} H_n(f, 1) = \frac{f(-1)}{3} + \lim_{n\to\infty} \frac{4}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1-x_k} - \lim_{n\to\infty} \frac{4}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{1+x_k},$$

which with the help of lemma 3 gives

$$\lim_{n\to\infty} H_n(f, 1) = f(1).$$

This proves theorem 1 for $x=1$. For $x=-1$ theorem 1 follows from continuity.

4. Quasi-Hermite-Fejér interpolation process $Q_n(f, x)$ defined by the conditions:

$$Q_n(f, 1) = f(1), \quad Q_n(f, -1) = f(-1),$$

$$Q_n(f, x_k) = f(x_k), \quad Q_n'(f, x_k) = 0, \quad k = 1, 2, \ldots$$

is given by

$$Q_n(f,x)=f(1)\frac{1+x}{2}\frac{W_n^2(x)}{W_n^2(1)}+f(-1)\frac{1-x}{2}\frac{W_n^2(x)}{W_n^2(-1)}+ \tag{22}$$

$$+\sum_{k=1}^{n}f(x_k)\frac{1-x^2}{1-x_k^2}\left[1+\left\{\frac{2x_k}{1-x_k^2}-\frac{W_n''(x_k)}{W_n'(x_k)}\right\}(x-x_k)\right]\left[\frac{W_n(x)}{W_n'(x_k)(x-x_k)}\right]^2.$$

One can easily prove the following.

Theorem 3. *The quasi-Hermite-Fejér interpolation polynomial $Q_n(f,x)$ given by* (22) *constructed on the zeros of Jacobi polynomial $P_n^{(-1/2,1/2)}(x)$ or $P_n^{(1/2,-1/2)}(x)$ converges uniformly to $f(x)$, when $f(x)$ is continuous in* $[-1,1]$.

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