Asymptotic properties of solutions of a second order nonlinear differential equation

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1. *Introduction*

Many physical systems are modeled by second order nonlinear differential equations of the type

$$(a(t)x')' + h(t, x, x') + q(t)f(x) = e(t, x, x')$$

where $h(t, x, x')$ represents a damping or frictional force and $e(t, x, x')$ represents an external force or perturbation of the system. In this paper we give sufficient conditions for solutions of the above equation to converge to zero. In so doing we generalize some results of HATVANI [2] and WILLETT and WONG [6] who studied the above equation when $h(t, x, x') \equiv e(t, x, x') \equiv 0$. We also include some continuability and boundedness theorems which extend results in [1—6]. None of the results in this paper explicitly require that the forcing term $e(t, x, x')$ be "small". For a discussion of problems related to the ones in this paper we refer the reader to [1—6] and the references contained therein.

2. *Asymptotic properties of solutions*

Consider the equation

$$(a(t)x')' + h(t, x, x') + q(t)f(x) = e(t, x, x') \tag{1}$$

where $q: [t_0, \infty) \to R$, $f: R \to R$, $h, e: [t_0, \infty) \times R^2 \to R$ are continuous, $a: [t_0, \infty) \to R$ is differentiable, $a(t) > 0$, $q(t) > 0$, and there are nonnegative continuous functions $r, w: [t_0, \infty) \to R$ such that

$$|e(t, x, y)| \leqq r(t)$$

and

$$-w(t)y^2 \leqq yh(t, x, y)$$

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for all (t, x, y) in $[t_0, \infty)\times R^2$. We will write equation (1) as the system

$$x' = y,$$

(2)

$$y' = (-a'(t)y - h(t, x, y) - q(t)f(x) + e(t, x, y))/a(t),$$

and make use of the following additional assumptions on the functions in (1):

$$\int_{t_0}^{\infty} \left[r(s)/a(s)\big(q(s)\big)^{1/2}\right] ds < \infty, \tag{3}$$

$$\int_{t_0}^{\infty} \left[r(s)/\big(q(s)\big)^{1/2}\right] ds < \infty, \tag{4}$$

$$\int_{t_0}^{\infty} [w(s)/a(s)]\, ds < \infty, \tag{5}$$

$$\int_{t_0}^{\infty} \left[\big(a(s)q(s)\big)'_-/a(s)q(s)\right] ds < \infty, \tag{6}$$

$$F(x) = \int_0^x f(s)\, ds \to \infty \quad \text{as} \quad |x| \to \infty, \tag{7}$$

$$\int_{t_0}^{\infty} [r(s)/a(s)]\, ds < \infty, \tag{8}$$

$$\int_{t_0}^{\infty} [r(s)/q(s)]\, ds < \infty \tag{9}$$

where $\big(a(t)q(t)\big)'_- = \max\left\{-\big(a(t)q(t)\big)', 0\right\}$.

Theorem 1. *If $F(x)$ is bounded from below, then all solutions of (2) can be defined for all $t \geqq t_0$.*

Proof. Suppose there is a solution $\big(x(t), y(t)\big)$ of (2) and $T > t_0$ such that $\lim_{t\to T^-} [|x(t)| + |y(t)|] = +\infty$. Since $F(x)$ is bounded from below, there exists $K > 0$ such that $F(x) > -K$ for all x. Defining $v(x, y, t) = a(t)y^2/q(t) + 2\big(F(x) + K\big)$ and letting $V(t) = v\big(x(t), y(t), t\big)$ we have

$$V' = 2a(t)yy'/q(t) + y^2\big(a(t)/q(t)\big)' + 2f(x)y =$$

$$= -\big(a(t)q(t)\big)'y^2/q^2(t) - 2h(t, x, y)y/q(t) + 2e(t, x, y)y/q(t) \leqq$$

$$\leqq y^2\big(a(t)q(t)\big)'_-/q^2(t) + 2w(t)y^2/q(t) + 2r(t)|y|/q(t).$$

Since

$$2|y|/\big(q(t)\big)^{1/2} \leqq y^2/q(t) + 1, \tag{10}$$

we have

$$V' \leqq y^2(a(t)q(t))'_-/q^2(t)+2w(t)y^2/q(t)+r(t)y^2/(q(t))^{3/2}+r(t)/(q(t))^{1/2} \leqq$$
$$\leqq [(a(t)q(t))'_-/a(t)q(t)+2w(t)/a(t)+r(t)/a(t)(q(t))^{1/2}]V+r(t)/(q(t))^{1/2}.$$

Integrating, we obtain

$$V(t) \leqq V(t_0)+$$
$$+\int_{t_0}^{t} [(a(s)q(s))'_-/a(s)q(s)+2w(s)/a(s)+r(s)/a(s)(q(s))^{1/2}]V(s)\,ds+$$
$$+\int_{t_0}^{t} [r(s)/(q(s))^{1/2}]\,ds.$$

Noticing that the second integral above is bounded on $[t_0, T]$ and applying Gronwall's inequality we have

$$V(t) \leqq K_1 \exp \int_{t_0}^{t} [(a(s)q(s))'_-/a(s)q(s)+2w(s)/a(s)+r(s)/a(s)(q(s))^{1/2}]\,ds \leqq K_2 < \infty$$

so $y^2(t) \leqq K_2 q(t)/a(t) \leqq K_3$ on $[t_0, T)$. This implies that $y(t)=x'(t)$ is bounded on $[t_0, T)$, and an integration yields that $x(t)$ is also bounded on $[t_0, T)$ contradicting the assumption that $(x(t), y(t))$ is a solution of (2) with finite escape time.

Remark 1. Theorem 1 generalizes continuability results in [2] and [5] as well as a special case of some results obtained by the authors in [4] for the equation $(a(t)x')'+q(t)f(x)g(x')=r(t)$.

Theorem 2. *If* (3)—(7) *hold, then all solutions of* (1) *are bounded.*

PROOF. First note that (7) implies that $F(x) > -K$ for some $K > 0$. Now define V as in the proof of Theorem 1 and differentiate to obtain

$$V' \leqq y^2(a(t)q(t))'_-/q^2(t)+2w(t)y^2/q(t)+2r(t)|y|/q(t).$$

Applying inequality (10), integrating, and using condition (4), we have

$$V(t) \leqq K_1+\int_{t_0}^{t} [(a(s)q(s))'_-/a(s)q(s)+2w(s)/a(s)+r(s)/a(s)(q(s))^{1/2}]V(s)\,ds.$$

It then follows from Gronwall's inequality and conditions (3) and (5)—(6) that $V(t)$ is bounded. Hence $F(x(t))$ is bounded and so $x(t)$ is bounded by (7).

Theorem 3. *If* (5)—(9) *hold, then all solutions of* (1) *are bounded.*

PROOF. Proceeding exactly as in the proof of Theorem 2 but replacing (10) by the inequality

$$(10)' \qquad 2|y| \leqq y^2+1,$$

we have

$$V' \leqq y^2(a(t)q(t))'_-/q^2(t)+2w(t)y^2/q(t)+r(t)y^2/q(t)+r(t)/q(t).$$

The remainder of the proof then follows as before.

Notice that Theorems 2 and 3 are independent of each other, for the equation

$$(t^3x')'+x^3/t = 1/t^2, \quad t>0$$

satisfies the hypotheses of Theorem 2 but (9) does not hold. On the other hand, the equation

$$(t^2x')'+t^2x^3 = 1, \quad t>0$$

satisfies Theorem 3 but (4) does not hold.

Remark 2. It follows from the proofs of Theorems 2 and 3 that if $q(t)/a(t)$ is bounded, then $y(t)=x'(t)$ is also bounded.

Remark 3. The boundedness results above improve work of Baker [1], Hatvani [2, 3], Mamii and Mirzov [5], Willett and Wong [6], and the present authors [4] as noted in Remark 1.

It will be convenient to classify solutions of (1) in the following way (see [4]). A solution $x(t)$ will be called nonoscillatory if there exists $t_1 \geqq t_0$ such that $x(t)\neq 0$ for $t\geqq t_1$; the solution will be called oscillatory if for any given $t_1\geqq t_0$ there exist t_2 and t_3 satisfying $t_1<t_2<t_3$, $x(t_2)>0$, and $x(t_3)<0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

To see that equations of the type (1) can have solutions possessing these various types of behavior, consider

$$x''+x = 1.$$

This equation has the nonoscillatory solution $x(t)=1+(1/2)\sin t$, the oscillatory solution $x(t)=1+2\sin t$, and the Z-type solution $x(t)=1+\sin t$.

The following lemma will be needed in the proof of Theorem 5.

Lemma 4. *Suppose there is a continuous function* $w_1\colon[t_0,\infty)\to R$ *such that* $|h(t,x,y)|\leqq w_1(t)$, $xf(x)>0$ *if* $x\neq 0, f(x)$ *is bounded away from zero if* x *is bounded away from zero, and*

$$\text{(11)} \qquad \int_{t_0}^{\infty}[N/a(s)]\,ds+\int_{t_0}^{\infty}[1/a(s)]\Big(\int_{t_0}^{s}[w_1(u)+r(u)-Mq(u)]\,du\Big)\,ds = -\infty$$

for all positive constants N *and* M. *If* $x(t)$ *is a nonoscillatory solution of* (1), *then* $\liminf_{t\to\infty}|x(t)|=0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t)>0$ for $t\geqq t_1\geqq t_0$ and assume that $\liminf_{t\to\infty} x(t)\neq 0$. Then there exists $t_2\geqq t_1$ such that $x(t)\geqq A>0$ for $t\geqq t_2$, so there exists $M>0$ such that $f(x(t))>M$ for $t\geqq t_2$. From (1) we have

$$\big(a(t)x'(t)\big)'\leqq w_1(t)+r(t)-Mq(t)$$

and integrating twice we obtain

$$x(t)\leqq x(t_2)+\int_{t_2}^{t}[a(t_2)x'(t_2)/a(s)]\,ds+$$

$$+\int_{t_2}^{t}[1/a(s)]\Big(\int_{t_2}^{s}[w_1(u)+r(u)-Mq(u)]\,du\Big)\,ds.$$

Hence by (11), $x(t)\to-\infty$ as $t\to\infty$ which is a contradiction. The proof in case $x(t)$ is ultimately negative is similar.

Theorem 5. *If, in addition to the hypotheses of Lemma* 4, *either* (3)—(7) *or* (5)—(9) *hold, then every nonoscillatory or Z-type solution* $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t)=0$.

PROOF. We offer a proof only for the case when (3)—(7) hold. The proof of the other case is similar and is left to the reader. Notice first that $xf(x)>0$ for $x\neq 0$ implies that $F(x)>0$ for $x\neq 0$, so begin as in the proof of Theorem 2 with $K=0$. Let $\varepsilon>0$ be given and let $x(t)$ be a nonoscillatory solution of (1) which is not ultimately monotonic. Then by conditions (3)—(6) and Lemma 4, there exists $t_1\geqq t_0$ such that

$$y(t_1)=0,\ F(x(t_1))<\varepsilon/4e^1,\ \int_{t_1}^{\infty}\left[r(s)/(q(s))^{1/2}\right]ds<\varepsilon/2e^1,$$

and

$$\int_{t_1}^{\infty}\left[(a(s)q(s))'_-/a(s)q(s)+2w(s)/a(s)+r(s)/a(s)(q(s))^{1/2}\right]ds<1.$$

Now if $x(t)$ is a Z-type solution, we could choose $y(t_1)=F(x(t_1))=0$, so in either case we have

$$V(t)\leqq[V(t_1)+\varepsilon/2e^1]\exp\int_{t_1}^{t}\left[(a(s)q(s))'_-/a(s)q(s)+\right.$$
$$\left.+2w(s)/a(s)+r(s)/a(s)(q(s))^{1/2}\right]ds\leqq[\varepsilon/e^1]\exp(1)=\varepsilon.$$

Thus $F(x(t))<\varepsilon$ for $t\geqq t_1$ so $F(x(t))\to 0$ as $t\to\infty$, which in turn implies that $x(t)\to 0$ as $t\to\infty$. To complete the proof we note that if $x(t)$ is a nonoscillatory solution of (1) which is ultimately monotonic, then by Lemma 4, $x(t)\to 0$ as $t\to\infty$.

Remark 4. Theorem 5 generalizes Theorem 2.2 in [2].

We note that condition (11) is not an unreasonable assumption in Theorem 5 since Hatvani [2; Theorem 2.1] showed that if

$$h(t,x,y)\equiv e(t,x,y)\equiv 0,\quad\text{then}\quad\int_{t_0}^{\infty}[1/a(s)]\Big(\int_{t_0}^{s}q(u)\,du\Big)ds=\infty$$

is a necessary condition for all solutions of (1) to converge to zero. The following theorem shows that condition (11) is "close" to being necessary for solutions of (1) to converge to zero. It includes the above mentioned result in [2] as a special case.

Theorem 6. *If, in addition to the hypotheses of either Theorem* 2 *or Theorem* 3, *we have*

$$\int_{t_0}^{\infty}[1/a(s)]\Big(\int_{t_0}^{s}[r(u)+w_1(u)]\,du\Big)ds<\infty$$

and

$$\int_{t_0}^{\infty}[1/a(s)]\Big(\int_{t_0}^{s}q(u)\,du\Big)ds<\infty,$$

then there is a solution $x(t)$ *of* (1) *such that* $\liminf_{t\to\infty}|x(t)|\neq 0$.

PROOF. From the proof of Theorem 2 or 3 we have

$$V'(t) \leqq k_1(t)V(t)+k_2(t)$$

where $k_1(t)$ and $k_2(t)$ are nonnegative continuous functions such that

$$\int_{t_0}^{\infty} k_1(s)\,ds \leqq P_1 < \infty \quad \text{and} \quad \int_{t_0}^{\infty} k_2(s)\,ds \leqq P_2 < \infty.$$

Hence if $x(t)$ is a solution of (1) such that $x(t_1)=1$, $y(t_1)=0$ for any $t_1 \geqq t_0$, then

$$V(t) \leqq V(t_1)+\int_{t_0}^{t} [k_1(s)V(s)+k_2(s)]\,ds$$

so

$$V(t) \leqq \left[2\big(F(1)+K\big)+P_2\right]\exp(P_1) = P_3 < \infty$$

for all $t \geqq t_1 \geqq t_0$. That is, $F(x(t)) \leqq P_3$ where P_3 is a constant which is independent of the choice of $t_1 \geqq t_0$. Therefore there exists $P_4 > 0$ such that $|x(t)| \leqq P_4$ for $t \geqq t_1$ and so there is a constant $A > 0$ such that $|f(x(t))| \leqq A$ for all $t \geqq t_1$.

Now choose $T \geqq t_0$ such that

$$\int_{T}^{\infty} [1/a(s)]\Big(\int_{T}^{s} [r(u)+w_1(u)]\,du\Big)\,ds < 1/4$$

and

$$\int_{T}^{\infty} [1/a(s)]\Big(\int_{T}^{s} q(u)\,du\Big)\,ds < 1/4A.$$

Let $z(t)$ be a solution of (1) such that $z(T)=1$ and $z'(T)=0$. Then

$$\big(a(t)z'(t)\big)' \geqq -r(t)-w_1(t)-Aq(t)$$

and integrating twice we obtain

$$z(t) \geqq 1-\int_{T}^{t} [1/a(s)]\Big(\int_{T}^{s} [r(u)+w_1(u)]\,du\Big)\,ds -$$

$$-A\int_{T}^{t} [1/a(s)]\Big(\int_{T}^{s} q(u)\,du\Big)\,ds > 1-1/4-1/4 = 1/2$$

for $t \geqq T$ and so $\liminf_{t\to\infty} |z(t)| \neq 0$.

In the final two theorems in this paper we will need the following conditions. Assume that

$$(12) \qquad \big(a(t)q(t)\big)' \geqq 0,$$

$$(13) \qquad cxf(x) \geqq 2F(x) > 0 \quad \text{if} \quad x \neq 0,$$

where c is a constant, and if $(x, y) \in M \times R$ where M is a bounded subset of R, then

$$(14) \qquad [e(t,x,y)-h(t,x,y)]/q(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Also, there exist nonnegative continuous functions $w_2, w_3\colon [t_0, \infty)\to R$ such that

(15) $$|h(t,x,y)y| \leqq w_2(t)y^2 + w_3(t)$$

for $(x, y)\in M\times R$, and

(16) $$\int_{t_0}^{\infty} [w_2(s)/a(s)]\,ds < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} [w_3(s)/q(s)]\,ds < \infty.$$

Theorem 7. *Suppose that conditions* (5), (7), (12)—(16), *and either* (3)—(4) *or* (8)—(9) *hold. If there exists a positive function* $d\colon [t_0, \infty)\to R$, $d\in C^3$ *such that*

$$d'(t) > 0 \quad \text{and} \quad d(t)\to\infty \quad \text{as} \quad t\to\infty,$$

(17) $$E = \liminf_{t\to\infty}\big[d(t)\big(a(t)q(t)\big)'/a(t)q(t)d'(t)\big] > c$$

and

$$\int_{t_0}^{t} \{[(d'(s)/q(s))'a(s)]'_-\}\,ds = o\big(d(t)\big) \quad \text{as} \quad t\to\infty,$$

then any oscillatory or Z-type solution $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. For $t\geqq u\geqq t_0$ let $V_u(t) = a(t)y^2/q(t) + 2F(x) +$

$$+2\int_{u}^{t}\big[h\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds - 2\int_{u}^{t}\big[e\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds.$$

Then $V_u'(t) = -\big(a(t)q(t)\big)'y^2/q^2(t)\leqq 0$, so $\lim_{t\to\infty} V_u(t) = R$ where possibly $R = -\infty$.

Let $x(t)$ be an oscillatory or Z-type solution of (1). By Theorem 2 or 3, $|x(t)|\leqq \leqq B$ and $|y(t)|\,(a(t))^{1/2}/(q(t))^{1/2}\leqq D$ for some costants B and D. Therefore from conditions (3)—(4) (respectively (8)—(9)) and by an application of the estimate (10) (respectively (10)′), we obtain

$$-2\int_{t_0}^{t}\big[e\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds < \infty.$$

Also, from (15)

$$\int_{t_0}^{t}\big[h\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds \leqq \int_{t_0}^{t}[D^2 w_2(s)/a(s) + w_3(s)/q(s)]\,ds < \infty.$$

Next, we will show that if $L>0$, then there exists $T\geqq t_0$ such that $\lim_{t\to\infty} V_u(t) < L$ for each $u\geqq T$. Let $L>0$ be given and let $K = E - c > 0$. Choose $T\geqq t_0$ such that

$$-2\int_{T}^{t}\big[e\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds < m,$$

$$2\int_{T}^{t}\big[h\big(s, x(s), y(s)\big)y(s)/q(s)\big]\,ds < m,$$

and

$$cx(t)\big[e\big(t, x(t), y(t)\big) - h\big(t, x(t), y(t)\big)\big]/q(t) < 2m$$

for $t \geqq T$ where $m = \min\{KL/32, L/32\}$. Let $k = K/2(2-K)$ if $K<2$, and $k=1$ otherwise. Now suppose there exists $u_0 \geqq T$ such that $V_{u_0}(t) \to P \geqq L$ as $t \to \infty$. Then there exists $T_1 \geqq u_0$ so that $P \leqq V_{u_0}(t) < (1+k)P$ for $t \geqq T_1$. Let

$$W(t) = d(t)V_{u_0}(t) + cd'(t)a(t)x(t)y(t)/q(t) - c(d'(t)/q(t))'a(t)x^2(t)/2.$$

Then

$$\begin{aligned} W' &= d(t)V'_{u_0}(t) + d'(t)V_{u_0}(t) + c[d'(t)a'(t)/q(t) + \\ &\quad + (d'(t)/q(t))'a(t)]x(t)y(t) + cd'(t)a(t)[x(t)y'(t) + \\ &\quad + y^2(t)]/q(t) - c[(d'(t)/q(t))'a(t)]'x^2(t)/2 - \\ &\quad - c(d'(t)/q(t))'a(t)x(t)y(t) = \\ &= -d(t)(a(t)q(t))'y^2(t)/q^2(t) + d'(t)\{a(t)y^2(t)/q(t) + \\ &\quad + 2F(x(t)) + 2\int_{u_0}^{t} [h(s, x(s), y(s))y(s)/q(s)]\,ds + \\ &\quad - 2\int_{u_0}^{t} [e(s, x(s), y(s))y(s)/q(s)]\,ds\} + \\ &\quad + cd'(t)a'(t)x(t)y(t)/q(t) + cd'(t)[x(t)e(t, x(t), y(t)) - \\ &\quad - a'(t)x(t)y(t) - x(t)h(t, x(t), y(t)) - \\ &\quad - x(t)q(t)f(x(t))]/q(t) + cd'(t)a(t)y^2(t)/q(t) - \\ &\quad - c[(d'(t)/q(t))'a(t)]'x^2(t)/2 \leqq \\ &\leqq d'(t)a(t)y^2(t)[(1+c) - d(t)(a(t)q(t))'/a(t)q(t)d'(t)]/q(t) + \\ &\quad + 2d'(t)\int_{u_0}^{t} \{[h(s, x(s), y(s)) - e(s, x(s), y(s))]y(s)/q(s)\}\,ds + \\ &\quad + 2md'(t) - c[(d'(t)/q(t))'a(t)]'x^2(t)/2. \end{aligned}$$

By (17), there exists $T_2 \geqq T_1$ such that

$$d(t)(a(t)q(t))'/a(t)q(t)d'(t) > (E+c)/2$$

for $t \geqq T_2$ and so

$$\begin{aligned} W' &\leqq d'(t)a(t)y^2(1-K/2)/q(t) + 2d'(t)\int_{u_0}^{t} \{[h(s, x(s), y(s)) - \\ &\quad - e(s, x(s), y(s))]y(s)/q(s)\}\,ds + 2md'(t) + K_1[(d'(t)/q(t))'a(t)]'_- \end{aligned}$$

for some constant $K_1 > 0$ and all $t \geqq T_2$. If $K \geqq 2$, then

$$W' \leqq 4md'(t) + K_1[(d'(t)/q(t))'a(t)]'_-$$

and if $K<2$,

$$W' \leqq d'(t)\Big[a(t)y^2(t)/q(t)+$$

$$+2\int_{u_0}^{t}\{[h(s,x(s),y(s))-e(s,x(s),y(s))]y(s)/q(s)\}\,ds\Big](1-K/2)+$$

$$+Kd'(t)\int_{u_0}^{t}\{[h(s,x(s),y(s))-e(s,x(s),y(s))]y(s)/q(s)\}\,ds+$$

$$+2md'(t)+K_1[(d'(t)/q(t))'a(t)]'_- \leqq$$

$$\leqq d'(t)(1-K/2)V_{u_0}(t)+(K/2)(2m)d'(t)+$$

$$+2md'(t)+K_1[(d'(t)/q(t))'a(t)]'_- <$$

$$<(1-K/2)(1+k)Pd'(t)+4md'(t)+K_1[(d'(t)/q(t))'a(t)]'_- \leqq$$

$$\leqq (1-K/4)Pd'(t)+KLd'(t)/8+K_1[(d'(t)/q(t))'a(t)]'_-.$$

Let $\{t_n\}$ be an increasing sequence of zeros of $x(t)$ such that $t_1\geqq T_2$ and $t_n\to\infty$ as $n\to\infty$. Integrating for the case $K<2$ we obtain

$$Pd(t_n)\leqq d(t_n)V_{u_0}(t_n)=W(t_n)\leqq$$

$$\leqq K_2+(1-K/4)Pd(t_n)+KLd(t_n)/8+K_1\int_{t_1}^{t_n}\{[(d'(s)/q(s))'a(s)]'_-\}\,ds$$

for each $n>1$. Since $P\geqq L$, we have

$$1\leqq K_2/Pd(t_n)+1-K/8+K_1\int_{t_1}^{t_n}\{[(d'(s)/q(s))'a(s)]'_-\}\,ds/Pd(t_n)$$

which yields a contradiction since $t_n\to\infty$ as $n\to\infty$. A similar contradiction is obtained if $K\geqq 2$.

To complete the proof of the theorem, let $\varepsilon>0$ be given and choose $T\geqq t_0$ such that $\lim_{t\to\infty} V_u(t)<\varepsilon/4$ for each $u\geqq T$. Choose $t_1\geqq T$ such that

$$2\int_{t_1}^{t}\{[e(s,x(s),y(s))-h(s,x(s),y(s))]y(s)/q(s)\}\,ds<\varepsilon/2$$

and

$$V_u(t)<\varepsilon/2$$

for $t\geqq t_1$. Then

$$2F(x(t))\leqq V_u(t)+\varepsilon/2<\varepsilon$$

for $t\geqq t_1$. Hence $F(x(t))\to 0$ and so $x(t)\to 0$ as $t\to\infty$.

Remark 5. Notice that it follows immediately from the last part of the proof that $(a(t)/q(t))^{1/2}y(t)\to 0$ as $t\to\infty$ and so Theorem 7 is a direct extension of Theorem 3.1 in [2].

Remark 6. If, in addition to the hypotheses of Theorem 7, we have $q(t)/a(t)$ bounded, then $y(t)$ would be bounded as was noted in Remark 2. In this case conditions (14) and (15) need only to hold for $(x, y)\in M\times N$ where M and N are bounded subsets of R.

Theorem 8. *Suppose that conditions* (5), (7), (12)—(16), *and either* (3)—(4) *or* (8)—(9) *hold. If there exists a positive continuous function* $b\colon [t_0, \infty)\to R$ *such that*

$$\int_{t_0}^{\infty} [1/b(s)]\, ds = \infty,$$

$$\liminf_{t\to\infty} \big[(a(t)q(t))' b(t)/a(t)q(t)\big] > 0,$$

and

$$\int_{t_0}^{t} \big[(a(s)/b(s))'_+/(a(s)q(s))^{1/2}\big]\, ds = o\Big(\int_{t_0}^{t} [1/b(s)]\, ds\Big) \quad \text{as} \quad t\to\infty, \tag{18}$$

then every solution $x(t)$ of (1) satisfies $\lim_{t\to\infty} x(t)=0$.

Proof. Let $x(t)$ be a solution of (1) and $\varepsilon>0$ be given. As in the proof of Theorem 7, $|x(t)|\leqq B$ and $|y(t)|\,(a(t))^{1/2}/(q(t))^{1/2}\leqq D$ for $t\geqq t_0$, so choose $t_1\geqq t_0$ such that

$$-2\int_{t_1}^{t} \big[e(s, x(s), y(s))y(s)/q(s)\big]\, ds < \varepsilon/8,$$

$$2\int_{t_1}^{t} \big[h(s, x(s), y(s))y(s)/q(s)\big]\, ds < \varepsilon/8,$$

and

$$cx(t)\big[e(t, x(t), y(t))-h(t, x(t), y(t))\big]/q(t) < \varepsilon/4$$

for $t\geqq t_1$. Define

$$V(t) = a(t)y^2(t)/q(t)+2F(x(t))+$$

$$+2\int_{t_1}^{t} \big[h(s, x(s), y(s))y(s)/q(s)\big]\, ds-2\int_{t_1}^{t} \big[e(s, x(s), y(s))y(s)/q(s)\big]\, ds;$$

then $V'(t)=-(a(t)q(t))'y^2(t)/q^2(t)\leqq 0$. Now let

$$W(t) = V(t)\int_{t_1}^{t} [1/b(s)]\, ds$$

and so

$$W'(t) = V(t)/b(t)-\big[(a(t)q(t))'y^2(t)/q^2(t)\big]\int_{t_1}^{t} [1/b(s)]\, ds.$$

Since for $t \geqq t_1$,

$$V(t) \leqq a(t)y^2(t)/q(t)+2F(x(t))+\varepsilon/4 =$$
$$= (1+c)a(t)y^2(t)/q(t)+2F(x(t))-cx(t)f(x(t))-$$
$$-c[a(t)x(t)y(t)]'/q(t)+cx(t)\big[e(t, x(t), y(t))-$$
$$-h(t, x(t), y(t))\big]/q(t)+\varepsilon/4 \leqq$$
$$\leqq (1+c)a(t)y^2(t)/q(t)-c[a(t)x(t)y(t)]'/q(t)+\varepsilon/2,$$

we have

$$W'(t) \leqq \Big\{1+c-\big[(a(t)q(t))'b(t)/a(t)q(t)\big] \int_{t_1}^{t} [1/b(s)]\,ds\Big\}\, a(t)y^2(t)/q(t)b(t)-$$
$$-c[a(t)x(t)y(t)]'/q(t)b(t)+\varepsilon/2b(t) \leqq -c[a(t)x(t)y(t)]'/q(t)b(t)+\varepsilon/2b(t)$$

for $t \geqq t_2$, for some $t_2 \geqq t_1$. Integrating, we have

$$W(t) \leqq W(t_2)-c\int_{t_2}^{t}\big[(a(s)x(s)y(s))'/q(s)b(s)\big]\,ds+(\varepsilon/2)\int_{t_2}^{t}[1/b(s)]\,ds.$$

An integration by parts yields

$$\int_{t_2}^{t}\big[(a(s)x(s)y(s))'/q(s)b(s)\big]\,ds =$$
$$= K_1+a(t)x(t)y(t)/q(t)b(t)+\int_{t_2}^{t}\big[(1/q(s)b(s))'\,a(s)x(s)y(s)\big]\,ds$$

where K_1 is a constant. Now

$$|a(t)x(t)y(t)| \leqq BD[a(t)q(t)]^{1/2}$$

so we have

$$c\Big|\int_{t_2}^{t}\big[(a(s)x(s)y(s))'/q(s)b(s)\big]\,ds\Big| \leqq c\,|K_1|+$$
$$+cBD\Big\{(a(t))^{1/2}/(q(t))^{1/2}b(t)+\int_{t_2}^{t}\big|(1/q(s)b(s))'\big|\,[a(s)q(s)]^{1/2}\,ds\Big\},$$

and thus

$$W(t) \leqq K_2+K_3\Big\{(a(t))^{1/2}/(q(t))^{1/2}b(t)+$$
$$+\int_{t_2}^{t}\big|(1/q(s)b(s))'\big|\,[a(s)q(s)]^{1/2}\,ds\Big\}+(\varepsilon/2)\int_{t_2}^{t}[1/b(s)]\,ds.$$

Since

$$\{[a(t)/b(t)]\,[1/a(t)q(t)]^{1/2}\}' =$$
$$= [a(t)/b(t)]'\,[1/a(t)q(t)]^{1/2}-[a(t)/b(t)][a(t)q(t)]'/2[a(t)q(t)]^{3/2} =$$
$$= \{[1/q(t)b(t)]\,[a(t)q(t)]^{1/2}\}' =$$
$$= [1/q(t)b(t)]'\,[a(t)q(t)]^{1/2}+[a(t)/b(t)]\,[a(t)q(t)]'/2[a(t)q(t)]^{3/2},$$

we have

$$|(1/q(t)b(t))'|\,[a(t)q(t)]^{1/2} \leqq$$
$$\leqq |(a(t)/b(t))'|/[a(t)q(t)]^{1/2} + a(t)\,[a(t)q(t)]'/[a(t)q(t)]^{3/2}b(t).$$

An integration by parts gives

$$\int_{t_2}^{t} \{a(s)[a(s)q(s)]'/[a(s)q(s)]^{3/2}b(s)\}\,ds =$$
$$= K_4 - 2(a(t))^{1/2}/(q(t))^{1/2}b(t) + 2\int_{t_2}^{t} \{[a(s)/b(s)]'/[a(s)q(s)]^{1/2}\}\,ds$$

so

$$W(t) \leqq K_2 + K_3\{(a(t))^{1/2}/(q(t))^{1/2}b(t) +$$
$$+ \int_{t_2}^{t} \{|(a(s)/b(s))'|/[a(s)q(s)]^{1/2}\}\,ds + K_4 - 2(a(t))^{1/2}/(q(t))^{1/2}b(t) +$$
$$+ 2\int_{t_2}^{t} \{[a(s)/b(s)]'/[a(s)q(s)]^{1/2}\}\,ds\} + (\varepsilon/2)\int_{t_2}^{t} [1/b(s)]\,ds \leqq$$
$$\leqq K_5 + K_3\int_{t_2}^{t} \{[|(a(s)/b(s))'| + 2[a(s)/b(s)]']/[a(s)q(s)]^{1/2}\}\,ds +$$
$$+ (\varepsilon/ \text{[illegible]} 2)\int_{t_2}^{t} [1/b(s)]\,ds.$$

Hence, by (18), there exists $T \geqq t_2$ such that $V(t) \leqq 3\varepsilon/4$ for $t \geqq T$. Thus

$$a(t)y^2(t)/q(t) + 2F(x(t)) \leqq 3\varepsilon/4 - 2\int_{t_1}^{t} [h(s, x(s), y(s))y(s)/q(s)]\,ds +$$
$$+ 2\int_{t_1}^{t} [e(s, x(s), y(s))y(s)/q(s)]\,ds < 3\varepsilon/4 + \varepsilon/8 + \varepsilon/8 = \varepsilon,$$

so $F(x(t)) \to 0$ as $t \to \infty$ and this implies that $x(t) \to 0$ as $t \to \infty$ completing the proof of the theorem.

Remark 7. Again it is easy to see from the last part of the proof that $(a(t)/q(t))^{1/2}y(t) \to 0$ as $t \to \infty$ and so Theorem 8 extends Theorem 1.1 of Willett and Wong [6]. The content of Remark 6 also applies to Theorem 8.

To see that Theorem 8 does not actually include Theorem 5, consider the equation

$$x'' + x = 1/t^2 + 6/t^4, \quad t > 0$$

whose general solution is

$$x(t) = A \sin t + B \cos t + 1/t^2.$$

The nonoscillatory solution of this equation converges to zero whereas the oscillatory solutions do not. Here Theorem 5 holds but Theorems 7 and 8 do not since $(a(t)q(t))' \equiv 0$.

Several intersting variations of Theorems 7 and 8 can be obtained by altering fome of the hypotheses of these theorems. For example, we can replace condition (14) by asking instead that $r(t)/q(t)\to 0$ as $t\to\infty$, there are nonnegative continuous sunctions $v_1, v_2\colon [t_0, \infty)\to R$ such that

$$|h(t, x, y)| \leqq v_1(t)|y| + v_2(t), \tag{19}$$

$v_2(t)/q(t)\to 0$ as $t\to\infty$, and either

i) $v_1(t)/(a(t)q(t))^{1/2} \to 0$,

ii) $v_1(t)/a(t)(q(t))^{1/2} \to 0$ and $v_1(t)/(q(t))^{1/2} \to 0$,

or,

iii) $v_1(t)/a(t) \to 0$ and $v_1(t)/q(t) \to 0$,

as $t\to\infty$. The three possibilities depend on whether we use that

$$(a(t))^{1/2}|y(t)|/(q(t))^{1/2} \leqq D,$$

inequality (10), or inequality (10)′ respectively after applying (19).

Other variations can be obtained by replacing (15) by (19). In this case we would need to replace (16) by $\int_{t_0}^{\infty} [v_1(s)/a(s)]\,ds < \infty$ and either

i) $\int_{t_0}^{\infty} \left[v_2(s)/(a(s)q(s))^{1/2}\right] ds < \infty$,

ii) $\int_{t_0}^{\infty} \left[v_2(s)/a(s)(q(s))^{1/2}\right] ds < \infty$ and $\int_{t_0}^{\infty} \left[v_2(s)/(q(s))^{1/2}\right] ds < \infty$,

or,

iii) $\int_{t_0}^{\infty} [v_2(s)/a(s)]\,ds < \infty$ and $\int_{t_0}^{\infty} [v_2(s)/q(s)]\,ds < \infty$

depending again on whether we use that $(a(t))^{1/2}|y(t)|/(q(t))^{1/2}$ is bounded, inequality (10) or inequality (10)′ respectively.

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