Degree of approximation by harmonic means of Fourier—Laguerre expansions

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1. The Fourier—Laguerre expansion of a function $f(x)\in L[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{1.1}$$

where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{(\alpha)}(y)\, dy \tag{1.2}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right), \tag{1.3}$$

and the existence of the integral in (1.2) is presumed.

A sequence $\{Sn\}$ is said to be summable by harmonic means if

$$\lim_{n\to\infty} (\log n)^{-1} \sum_{k=0}^{n} \frac{S_{n-k}}{k+1} \tag{1.4}$$

exists.

By the relation $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$, we have

$$\begin{aligned} \sum_{\nu=0}^{n} a_\nu L_\nu^{(\alpha)}(0) &= \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) \sum_{\nu=0}^{n} L_\nu^{(\alpha)}(y)\, dy = \\ &= \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y) L_n^{(\alpha+1)}(y)\, dy. \end{aligned} \tag{1.5}$$

The Cesàro summability of the series (1.1) at the point $x=0$ has been studied by KOGBETLIANTZ ([1]) and SZEGŐ ([4], [5]). Very recently, GUPTA ([2]) estimated the order of the function by Cesàro means of the series (1.1) at the point $x=0$ after replacing the continuity condition in Szegő's theorem ([5]) by a much lighter condition. He established the following theorems:

Theorem A: If

(1.6)
$$F(t)=\int_t^{\delta}\frac{|f(y)|}{y}\,dy=o\left(\log\frac{1}{t}\right)$$

and

(1.7)
$$\int_1^{\infty}e^{-y/2}y^{\alpha-k-\frac{1}{3}}|f(y)|\,dy<\infty,$$

then

$$\sigma_n^{(k)}(o)=o(\log n),$$

provided $k>\alpha+\dfrac{1}{2}$.

Theorem B: If

(1.8)
$$\int_t^{\delta}\frac{|f(y)|}{y}\,dy=o\left[\left(\log\frac{1}{t}\right)^{p+1}\right]$$

$$(t\to o,\ -1<p<\infty),$$

and if

$$\int_1^{\infty}e^{-y/2}y^{\alpha-k-\frac{1}{3}}|f(y)|\,dy<\infty,$$

then

$$\sigma_n^{(k)}(o)=o[(\log n)^{p+1}],$$

provided that $k>\alpha+1/2$.

PANDEY ([3]) investigated the harmonic summability of the series (1.1) at the point $x=0$ and proved the following theorem:

Theorem C: For $-\dfrac{5}{6}\leqq\alpha\leqq-\dfrac{1}{2}$, *the series* (1.1) *is summable by harmonic means at the point* $x=0$ *to the sum A, provided*

(1.9)
$$\int_0^{t}|\Phi(y)|\,dy=o(t^{\alpha+1}),\quad\text{as}\quad t\to o$$

(1.10)
$$\int_{\omega}^{n}e^{y/2}y^{-\frac{\alpha}{2}-\frac{3}{4}}|\Phi(y)|\,dy=o\left(n^{-\frac{\alpha}{2}-\frac{1}{4}}\right)$$

and

(1.11)
$$\int_n^{\infty}e^{y/2}y^{-\frac{1}{3}}|\Phi(y)|\,dy=o(1),$$

where

$$\Phi(y)=\{\Gamma(\alpha+1)\}^{-1}e^{-y}y^{\alpha}[f(y)-A].$$

2. The object of this paper is to estimate the order of the function by harmonic means of the series (1.1) at the point $x=0$. All the conditions of our theorem I are much lighter than the corresponding conditions of Pandey, but the condition (2.1) is the same as that of Gupta while (2.2) and (2.3) together are much lighter than

Gupta's second condition. In theorem II we have proved an extension of theorem I, like that of Gupta, by introducing a parameter p to arrive at a deeper insight into the behaviour of the harmonic means.

We write

$$\Phi(y) = \{\Gamma(\alpha+1)\}^{-1} e^{-y} y^{\alpha} [f(y) - f(0)]$$

and establish the following theorems:

Theorem I. *For* $-\frac{5}{6} \leqq \alpha \leqq -\frac{1}{2}$

$$t_n(0) - f(0) = o(\log n),$$

provided that

$$\int_t^{\omega} \frac{|\Phi(y)|}{y^{\alpha+1}}\, dy = o\left(\log \frac{1}{t}\right), \quad \text{as} \quad t \to 0, \tag{2.1}$$

ω fixed positive constant

$$\int_{\omega}^{n} e^{y/2} y^{-\frac{\alpha}{2}-\frac{3}{4}} |\Phi(y)|\, dy = o\left(n^{-\frac{\alpha}{2}-\frac{1}{4}} \log n\right) \tag{2.2}$$

and

$$\int_n^{\infty} e^{y/2} y^{-\frac{1}{3}} |\Phi(y)|\, dy = o(\log n), \quad \textit{as} \quad n \to \infty. \tag{2.3}$$

Theorem II. For $-\frac{5}{6} \leqq \alpha \leqq -\frac{1}{2}$

$$t_n(0) - f(0) = o[(\log n)^{p+1}],$$

provided that

$$\int_t^{\omega} \frac{|\Phi(y)|}{y^{\alpha+1}}\, dy = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right] \tag{2.4}$$

$$(\text{as } t \to 0, \ -1 < p < \infty);$$

ω fixed positive constant

$$\int_{\omega}^{n} e^{y/2} y^{-\frac{\alpha}{2}-\frac{3}{4}} |\Phi(y)|\, dy = o\left[n^{-\frac{\alpha}{2}-\frac{1}{4}} (\log n)^{p+1}\right] \tag{2.5}$$

and

$$\int_n^{\infty} e^{y/2} y^{-\frac{1}{3}} |\Phi(y)|\, dy = o[(\log n)^{p+1}], \quad \textit{as} \quad n \to \infty. \tag{2.6}$$

3. In the proof of the theorem we shall require the order estimates and the asymptotic properties of the Laguerre functions given by Szegő ([5], pp. 175 and 238).

Order estimates. If α is an arbitrary real number, nad c and ω are fixed positive constants, and $n \to \infty$, then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\frac{\alpha}{2}-\frac{1}{4}} O\left(n^{\frac{\alpha}{2}-\frac{1}{4}}\right), & \text{if} \quad \dfrac{c}{n} \leqq x \leqq \omega; \\ O(n^\alpha), & \text{if} \quad 0 \leqq x \leqq \dfrac{c}{n}. \end{cases} \tag{3.1}$$

Asymptotic property. If α be an arbitrary real number, $\omega > 0$, $0 < \eta < 4$, then for $n \to \infty$, we have

$$\max e^{-x/2} x^{\frac{\alpha}{2}+\frac{1}{4}} |L_n^{(\alpha)}(x)| \sim \begin{cases} n^{\frac{\alpha}{2}-\frac{1}{4}}, & \omega \leqq x \leqq (4-\eta)n; \\ n^{\frac{\alpha}{2}-\frac{1}{12}}, & x \geqq \omega. \end{cases} \tag{3.2}$$

4. Proof of theorem I. There is no loss of generality if we assume that $f(0)=0$. From (1.4) and (1.5), we have

$$t_n(0) = (\log n)^{-1} \sum_{k=0}^{n} \frac{1}{n-k+1} \int_0^\infty \Phi(y) L_k^{(\alpha+1)}(y)\, dy = \tag{4.1}$$

$$= (\log n)^{-1} \sum_{k=1}^{n} \frac{1}{n-k+1} \int_0^\infty \Phi(y) L_k^{(\alpha+1)}(y)\, dy +$$

$$+ (\log n)^{-1} \frac{1}{n+1} \int_0^\infty e^{-y} y^\alpha f(y) L_0^{(\alpha+1)}(y)\, dy =$$

$$= (\log n)^{-1} \sum_{k=1}^{n} \frac{1}{n-k+1} \left(\int_0^{c/n} + \int_{c/n}^{\omega} + \int_{\omega}^{n} + \int_{n}^{\infty} \right) \Phi(y) L_k^{(\alpha+1)}(y)\, dy + o(1) =$$

$$= I_1 + I_2 + I_3 + I_4 + o(1),$$

say, where ω is a fixed positive constant. Using (3.1) and (2.1), we get

$$I_1 = (\log n)^{-1} \sum_{k=1}^{n} \frac{O(k^{\alpha+1})}{n-k+1} \int_0^{c/n} |\Phi(y)|\, dy = O(n^{\alpha+1}) o(n^{-\alpha-1} \log n) = o(\log n). \tag{4.2}$$

Again, using (3.1), we find that

$$I_2 = (\log n)^{-1} \sum_{k=1}^{n} \frac{O\left(k^{\frac{\alpha}{2}+\frac{1}{4}}\right)}{n-k+1} \int_{c/n}^{\omega} |\Phi(y)| y^{-\frac{\alpha}{2}-\frac{3}{4}}\, dy =$$

$$= (\log n)^{-1} \sum_{k=1}^{n} \frac{O\left(k^{\frac{\alpha}{2}+\frac{5}{4}}\right)}{k(n-k+1)} \int_{c/n}^{\omega} y^{\frac{\alpha}{2}+\frac{1}{4}} \frac{|\Phi(y)|}{y^{\alpha+1}}\, dy =$$

$$= O\left(n^{\frac{\alpha}{2}+\frac{1}{4}} n^{-\frac{\alpha}{2}-\frac{1}{4}}\right) \int_{c/n}^{\omega} \frac{|\Phi(y)|}{y^{\alpha+1}}\, dy = o(\log n). \tag{4.3}$$

Now, using (3.2) and (2.2), we get

$$I_3 = (\log n)^{-1} \sum_{k=1}^{n} \frac{O(k^{\frac{\alpha}{2}+\frac{1}{4}})}{n-k+1} \int_{\omega}^{n} e^{y/2} y^{-\frac{\alpha}{2}-\frac{3}{4}} |\Phi(y)|\, dy =$$

$$= O\left[(\log n)^{-1} n^{\frac{\alpha}{2}+\frac{5}{4}}\right] \sum_{k=1}^{n} \frac{1}{k(n-k+1)} \int_{\omega}^{n} e^{y/2} y^{-\frac{\alpha}{2}-\frac{3}{4}} |\Phi(y)|\, dy =$$

$$= O\left(n^{\frac{\alpha}{2}+\frac{1}{4}}\right) o\left(n^{-\frac{\alpha}{2}-\frac{1}{4}} \log n\right) = o(\log n). \tag{4.4}$$

Finally, by second part of (3.2) and (2.3), we get

$$I_4 = (\log n)^{-1} \sum_{k=1}^{n} \frac{O(k^{\frac{\alpha}{2}+\frac{5}{12}})}{n-k+1} \int_{n}^{\infty} e^{y/2} |\Phi(y)|\, y^{-\frac{\alpha}{2}-\frac{3}{4}} dy =$$

$$= O\left(n^{\frac{\alpha}{2}+\frac{5}{12}}\right) \int_{n}^{\infty} e^{y/2} |\Phi(y)| \frac{y^{-\frac{1}{3}}}{y^{\frac{\alpha}{2}+\frac{5}{12}}}\, dy =$$

$$= O(1) \int_{n}^{\infty} e^{y/2} |\Phi(y)|\, y^{-\frac{1}{3}} dy = o(\log n). \tag{4.5}$$

Thus the theorem gets proved on account of (4.1), ..., (4.5).

5. Proof of theorem II. Theorem II can be proved exactly on the same lines as Theorem I.

I am highly grateful to Dr. D. P. Gupta for his valuable help during the preparation of this paper.

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(Received August 13, 1974.)