A method for obtaining proper classes of short exact sequences of Abelian groups

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1. Introduction

Let $\mathscr{C}$ be a class of short exact sequences of abelian groups. (Henceforth the word group will mean abelian group.) Let the notation $A \lhd B$ mean that the sequence $0 \to A \to B \to B/A \to 0$ is in $\mathscr{C}$. $\mathscr{C}$ is said to be a *proper class* of sequences if the following six properties are satisfied:

(P1) A sequence that is equivalent to an element of $\mathscr{C}$ is in $\mathscr{C}$.
(P2) If A is a direct summand of B, then $A \lhd B$.
(P3) If $A \lhd B$ and $A \subset H \subset B$, then $A \lhd H$.
(P4) If $A \lhd B$ and $K \subset A$, then $A/K \lhd B/K$.
(P5) If $A \lhd H$ and $H \lhd B$, then $A \lhd B$.
(P6) If $K \lhd B$ and $A/K \lhd B/K$, then $A \lhd B$.

An exposition of the relative homological algebra associated with a proper class may be found in Ref. [4] and Ref. [6]. The projectives and injectives relative to various proper classes have proved to be of considerable interest. In the category of abelian p-groups, for example, the totally projective groups are found to arise as the projectives relative to the proper class of *balanced* sequences (see Ref. [3]).

The purpose of this paper is to present a method for constructing proper classes of short exact sequences of groups by means of functorial topologies on the category of abelian groups. The precise formulation is given in section three. An illustration of the results of section three is given in section four, where we apply some of the results of Ref. [5] to compute the projectives and injectives for the proper class determined by the Z-adic topology.

The notation and terminology used here will be almost entirely that of Ref. [2].

2. Functorial topologies and imbedded subgroups

For each group A let $t(A)$ be a topology on A relative to which A is a topological group. Let $t^*(A)$ denote the set of open subgroups of A. Then we say that t is a *linear functorial* topology on the category of Abelian groups provided that

(1) $t^*(A)$ is a neighborhood basis of zero for every group A; and

(2) Every homomorphism, $f: A \to B$, is continuous relative to $t(A)$ and $t(B)$. The notion of a functorial topology is due to CHARLES [1].

A general method of obtaining linear functorial topologies is mentioned in Ref. [2]. Let X be any class of groups. For each group A let $X(A)$ be the set of subgroups of A which are finite intersections of the subgroups in the set $\{H \subset A | A/H \in X\}$. The X-topology, $t_x(A)$, on A is defined to be the smallest topology containing the subgroups in $X(A)$ and their cosets. It is easy to see that A is a topological group relative to the X-topology and that $X(A)$ is a neighborhood basis of zero. It is also easily shown that t_x is a linear functorial topology if and only if X is closed under subgroups (that is, each subgroup of A is in X if A is in X). It may be shown that the class of finite direct sums of elements of X determines the same topology as X for each group. Moreover, if X and Y are distinct classes of groups and are closed under subgroups and finite direct sums, then t_x and t_y are distinct linear functorial topologies on the category of Abelian groups. In the sequel we shall be concerned with linear functorial topologies relative to which each group epimorphism is an open function. We mention that t_x has this property if and only if X is closed under epimorphic images (provided, of course, that X is closed also under subgroups and finite direct sums). The most familiar linear functorial topologies are, perhaps, the Z-adic and finite index topologies. These are both X-topologies, the former determined by the class of bounded groups and the later determined by the class of finite groups. It should be noted that not every linear functorial topology is an X-topology.

Definition 2.1. *Let t be a linear functorial topology, B a group and A a subgroup of B. Let $t(B, A)$ denote the relative topology on A induced by $t(B)$. If $t(A) = t(B, A)$, then we shall say that A is imbedded in B with respect to t (or, simply, t-imbedded in B).*

It is clear that the set $t^*(B, A)$ of subgroups of A that are open relative to $t(B, A)$ is precisely $\{A \cap H | H \in t^*(B)\}$ and that $t^*(B, A)$ is a neighborhood basis of zero. Moreover, since the inclusion homorphism $\iota: A \to B$ is continuous and $\iota^{-1}(H) = A \cap H$, it follows that $A \cap H \in t^*(A)$ if $H \in t^*(B)$. Thus $t^*(B, A) \subset t^*(A)$. Hence A will be t-imbedded in B whenever $t^*(A) \subset t^*(B, A)$. This leads to the following obvious criterion.

Proposition 2.2. *The subgroup A is t-imbedded in B if and only if there exists a function $\Psi: t^*(A) \to t^*(B)$ satisfying $A \cap \Psi(K) \subset K$ for each $K \in t^*(A)$.* (We shall say that Ψ is a t-imbedding function for A in B.)

A short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is said to be t-imbedded if $\alpha(A)$ is a t-imbedded subgroup of B. The class of t-imbedded sequences is denoted by $\mathscr{C}_t$. We shall see in the next section that $\mathscr{C}_t$ is frequently a proper class of short exact sequences.

3. Proper classes of imbedded sequences

We are now ready for the main result of this paper. The proof consists of the two lemmas that follows the comments beneath the theorem.

Theorem 3.1. *Let* t *be a linear functorial topology relative to which each group epimorphism is an open function. Then the class* $\mathscr{C}_t$ *of* t*-imbedded sequences is a proper class of sequences.*

As was indicated in section two, there is an abundance of topologies in which epimorphisms are open. It should be noted, however, that the condition that epimorphism be open is not necessary for $\mathscr{C}_t$ to be a proper class. For example, let X_0 be the class of torsion free groups and $\mathscr{C}_0$ the class of short exact sequences that are imbedded relative to the X_0-topology. X_0 is not closed under epimorphic images, so group epimorphisms are not open, in general. It is easy to show, however, that $\mathscr{C}_0$ contains every short exact sequence. We do have an example of a linear functorial topology (not given here) for which properties (P4) and (P6) fail to hold. A precise necessary and sufficient condition for $\mathscr{C}_t$ to be a proper class is not known by the authors.

Lemma 3.2. *Let* t *be any linear functorial topology. Then* $\mathscr{C}_t$ *satisfies properties (P1), (P2), (P3), and (P5) for a proper class.*

Proof. If E_1 and E_2 are equivalent sequences, then we have the following commuting diagram:

$$\begin{array}{l} E_1\colon 0 \to A \to B_1 \to C \to 0 \\ \qquad\qquad\ \downarrow \quad\ \downarrow\theta \\ E_2\colon 0 \to A \to B_2 \to C \to 0. \end{array}$$

In each sequence we are regarding A as a subgroup of B_i. Let $E_1 \in \mathscr{C}_t$ with t-imbedding function Ψ_1. Since homomorphisms are continuous, then group isomorphisms are, in fact, homeomorphisms. Thus one may readily verify that a t-imbedding function for E_2 is the function $\Psi_2\colon t^*(A) \to t^*(B_2)$ defined by $\Psi_2(K) = (\theta\Psi_1\theta^{-1})(K)$ for every $K \in t^*(A)$. This verifies (P1).

Next let A be a direct summand of B and let $\pi\colon B \to A$ be the projection. One may readily verify that the function $\Psi\colon t^*(A) \to t^*(B)$ defined by $\Psi(K) = \pi^{-1}(K)$ for each $K \in t^*(A)$ is a t-imbedding function for A in B. This verifies (P2).

Next let Ψ_1 be a t-imbedding function for A in B and let $A \subset H \subset B$. Since $H \cap J \in t^*(H)$ if $J \in t^*(B)$, it follows that the function $\Psi_2\colon t^*(A) \to t^*(H)$ defined by $\Psi_2(K) = H \cap \Psi_1(K)$ is a t-imbedding function for A in H. This verifies (P3).

Finally let Ψ_1 and Ψ_2 be a t-imbedding functions for A in H and H in B, respectively. Define $\Psi\colon t^*(A) \to t^*(B)$ by $\Psi(K) = (\Psi_2\Psi_1)(K)$. Then

$$A \cap \Psi(K) = A \cap H \cap \Psi_2(\Psi_1(K)) \subset A \cap \Psi_1(K) \subset K.$$

Thus Ψ is a t-imbedding function for A in B, as desired. This verifies (P5).

Lemma 3.3. *Let* t *be a linear functorial topology relative to which each group epimorphism is open. Then* $\mathscr{C}_t$ *satisfies properties (P4) and (P6) for a proper class.*

PROOF. Let Ψ_1 be a t-imbedding function for A in B and let $K \subset A$. Let $\Phi_1 : A \to A/K$ and $\Phi_2 : B \to B/K$ be the canonical epimorphisms. Let $J/K \in t^*(A/K)$. Then $J = \Phi_1^{-1}(J/K) \in t^*(A)$ and $\Psi_1(J) \in t^*(B)$. Thus $(\Psi_1(J)+K)/K = \Phi_2(\Psi_1(J)) \in t^*(B/K)$, since epimorphisms are open relative to t. Let us define $\Psi : t^*(A/K) \to t^*(B/K)$ by $\Psi(J/K) = (\Psi_1(J)+K)/K$. Then Ψ is, indeed, a t-imbedding function for A/K in B/K, since

$$(A/K) \cap \Psi(J/K) = (A/K) \cap [(\Psi_1(J)+K)/K] = (A \cap \Psi_1(J)+K)/K \subset J/K.$$

This verifies (P4).

Now let Ψ_1 and Ψ_2 be t-imbedding functions for K in B and A/K in B/K, respectively. For each $J \in t^*(A)$ we wish to define $\Psi(J) \in t^*(B)$ such that $A \cap \Psi(J) \subset J$. Now $K \cap J \in t^*(K)$; so $\Psi_1(K \cap J) \in t^*(B)$. Then $J_1 = \Psi_1(K \cap J) \cap J$ is in $t^*(A)$, and we note that $(J_1+K) \cap \Psi_1(J \cap K) \subset J$. Also $(J_1+K)/K$ is in $t^*(A/K)$, since epimorphisms are open. Now let $\Psi_2((J_1+K)/K) = L/K$. Then $L \in t^*(B)$, so $L \cap \Psi_1(J \cap K) \in t^*(B)$. We now claim that $\Psi(J) = L \cap \Psi_1(J \cap K)$ works. First note that $(A \cap L)/K = (A/K) \cap (L/K) \subset (J_1+K)/K$, so that $A \cap L \subset J_1 + K$. Thus $A \cap \Psi(J) = A \cap (L \cap \Psi_1(J \cap K)) \subset (J_1+K) \cap \Psi_1(J \cap K) \subset J$, as desired. This verifies (P6).

4. The Z-adic projectives and injectives

Let us review some basic concepts. A group G is projective relative to the short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ provided that for each homomorphism $f : G \to C$ there is a homomorphism $\bar{f} : G \to B$ satisfying $b\bar{f} = f$. Injectivity is defined dually. If $\mathscr{C}$ is a proper class of sequences, then the class of projectives, $\mathscr{P}(\mathscr{C})$, associated with $\mathscr{C}$ is the class of groups that are projective relative to every sequence in $\mathscr{C}$. The injectives, $\mathscr{D}(\mathscr{C})$, are defined similarly. If $\mathscr{C}_1$ is the proper class of pure-exact sequences, then $\mathscr{P}(\mathscr{C}_1)$ is precisely the class of all direct sums of cyclic groups (see Ref. [2]). We need the following elementary result.

Lemma 4.1. *Let E_n denote the short exact sequence $0 \to nZ \xrightarrow{\iota} Z \to Z/nZ \to 0$, where Z is the infinite cyclic group of integers, n a positive integer and ι the inclusion map.*

(1) *If G is injective relative to E_n, then $nG = G$.*

(2) *If G is projective relative to E_n and $G \in \mathscr{P}(\mathscr{C}_1)$, then $G[n] = 0$, where $G[n] = \{g \in G \mid ng = 0\}$.*

PROOF. Let G be injective relative to E_n. Let $g \in G$ and let $\theta : nZ \to G$ be the homomorphism defined by $\theta(nk) = kg$ for each $k \in Z$. Then there exists $\bar{\theta} : Z \to G$ satisfying $\bar{\theta}(nk) = \theta(nk)$ for each $k \in Z$. Thus $g = \theta(n) = \bar{\theta}(n) = n\bar{\theta}(1)$; so $g \in nG$, as desired.

Next suppose $G \in \mathscr{P}(\mathscr{C}_1)$ and that $G[n] \neq 0$. Then G has a finite cyclic summand $\langle g \rangle$ whose order divides n. Thus there exists an epimorphism $f : G \to Z/nZ$ satisfying $f(g) \neq 0$. But every homomorphism $\bar{f} : G \to Z$ takes g to zero. Hence G is not projective relative to E_n, as desired.

According to the discussion in section two, the Z-adic topology is a linear functorial topology relative to which group epimorphisms are open. Thus the class $\mathscr{C}_z$ of Z-imbedded short exact sequences is a proper class of sequences. We wish to determine $\mathscr{P}(\mathscr{C}_z)$ and $\mathscr{D}(\mathscr{C}_z)$. One may easily check that for each group G, the set $\{nG|n$ a positive integer$\}$ is a neighborhood basis for zero for the Z-adic topology. Thus we have the following modification of Proposition 2.2 for the Z-adic topology.

Proposition 4.2 *Let N denote the positive integers. A subgroup A is Z-imbedded in a group B if and only if there exists a function $\Psi: N\to N$ satisfying $A\cap \cap\Psi(m)B\subset mA$ for every $m\in N$.*

Theorem 4. 3. *$\mathscr{P}(\mathscr{C}_z)$ is the class of free groups and $\mathscr{D}(\mathscr{C}_z)$ is the class of divisible groups.*

PROOF. First we observe that $E_n\in\mathscr{C}_z$ for every $n\in N$, since the function $\Psi(m)= =m+n$ serves as a Z-imbedding function for nZ in Z. Thus each member of $\mathscr{D}(\mathscr{C}_z)$ is divisible, by lemma 4.1. Clearly each divisible group is in $\mathscr{D}(\mathscr{C}_z)$. Next we observe that $\Psi(m)=m$ is a Z-imbedding function for each pure subgroup. Thus $\mathscr{C}_1\subset\mathscr{C}_z$; so $\mathscr{P}(\mathscr{C}_z)\subset\mathscr{P}(\mathscr{C}_1)$. Thus each member of $\mathscr{P}(\mathscr{C}_z)$ is torsion free and, hence, free. Conversely, every free group is in $\mathscr{P}(\mathscr{C}_z)$. This proves the theorem.

Let $\mathscr{E}$ denote the class of all short exact sequences. It follows from Lemma 4.4 below that $\mathscr{C}_z$ is strictly contained in $\mathscr{E}$. But $\mathscr{P}(\mathscr{E})=\mathscr{P}(\mathscr{C}_z)$ and $\mathscr{D}(\mathscr{E})=\mathscr{D}(\mathscr{C}_z)$. Thus, in the terminology of Ref. [6], $\mathscr{C}_z$ is neither projectively nor injectively closed.

Finally we observe that $\mathscr{C}_z$ contains neither enough projectives nor enough injectives. This observation is a consequence of lemma 4.4 whose proof may be found in Ref. [5].

Lemma 4.4. *Let E denote a short exact sequence.* $0\to A\to B\to C\to 0$.

(1) *If B is torsion free, then $E\in\mathscr{C}_z$ if and only if each p-primary component of C is bounded.*

(2) *If B is divisible, then $E\in\mathscr{C}_z$ if and only if A is divisible.*

Thus, in particular, only the divisible groups have injective resolutions and only the groups with bounded primary components have projective resolutions.

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(Received August 16, 1974.)