The Lie derivatives and areal motion in areal space

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Summary: The purpose of the present paper is to obtain the Lie derivatives of a vector field and connection coefficient in areal space. The Lie derivative thus obtained are used to obtain the areal motion and various commutation formulae.

1. *Introduction.* A space in which the fundamental metric concept i.e. an m-dimensional area is defined by means of an m-fold integral, is called an areal space. There are two approaches to introduce the geometrical concept of areal space. The first approach is due to DAVIES ([2], [3], [4], [5]), KAWAGUCHI ([6], [7]), KAWAGUCHI and KATSURADA ([8], [9]), KAWAGUCHI and TANDAI [10] and TANDAI ([11], [12], [13,] [14]). In these works a simple m-vector $\pi_{i_1 \dots i_m}$ satisfying the well known Plücker relations has been used. The other approach is due to RUND ([17],[18]) in which the use of this m-vector is avoided. This theory is based on a systematic exploitation of the homogeneity conditions which characterize Lagrangians of parameter invariant integrals.

The deformation theories in Finsler and Cartan spaces were developed by KNEBELMAN [1], DAVIES [2], RUND [16] and YANO [15]. These theories have also been studied in an areal space by IGARASHI [19]. The geometric concept in this work is based on the first approach of areal space. The purpose of this paper is to investigate Lie derivatives in areal spaces from the point of view of Rund's approach. After giving the fundamental formulae of areal spaces in § 2 we define the Lie derivative of a vector field in § 3. The section 4 is devoted for rewriting the Lie derivative of a vector field in terms of covariant partial derivatives. In § 5 the Lie derivative of the connection coefficient has been obtained. The various commutation formulae involving the Lie derivative have also been obteined. In the last section the concept of the areal motion has been introduced.

Our considerations are purely local in character. In some places the detailed calculations have been suppressed for the sake of brevity. With regard to such instances reference is made to RUND ([17], [18]).

Throughout this paper the Latin indices $i, j, h, k\dots$ run over 1 to n while Greek indices $\alpha, \beta, \gamma \dots$ run over 1 to m.

2. *Fundamental formulae.* Let a subspace C_m given by the equation

$$x^i = x^i(t^\alpha) \tag{2.1}$$

be immersed in an n-dimensional differentiable manifold X_n where t^α denotes a system of independent parameters in C_m. Assuming that the functions (2.1) are of class C^4, their first derivatives being denoted by

$$\dot{x}^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}. \tag{2.2}$$

The matrix $\left\| \frac{\partial x^i}{\partial t^\alpha} \right\|$ is always supposed to be of rank m.

We consider a Lagrange function $L(x^i, \dot{x}^i_\alpha)$ of the $n+nm$ variables $x^i, \dot{x}^i_\alpha$ satisfying the conditions

(A) The function L is of class C^4 in all its arguments and it is scalar with respect to transformations of the local coordinates x^i of X_n.

(B) The function L is positive for all independent sets of arguments $\dot{x}^i_\alpha$

(C) The integral

$$I = \int_G L(x^i, \dot{x}^i_\alpha)\, dt^1 \wedge \ldots \wedge dt^m \tag{2.3}$$

over a fixed region G of C_m is invariant under the transformations of the parameters t^α.

(D) The $nm \times nm$ determinant

$$D = \det \left[\frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha \partial \dot{x}^j_\beta} \right]$$

is non vanishing for linearly independent $\dot{x}^i_\alpha$.

The condition C is equivalent to the relation (Rund [17]).

$$\frac{\partial L}{\partial \dot{x}^i_\alpha} \dot{x}^i_\beta = \delta^\alpha_\beta L. \tag{2.4}$$

In view of (2.4) we have

$$mL^{2/m} = g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\varepsilon)\dot{x}^i_\alpha \dot{x}^j_\beta \tag{2.5}$$

where

$$g^{\alpha\beta}_{ij} = \frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha \partial \dot{x}^j_\beta}. \tag{2.6}$$

From (2.5) it is evident that if L is interpreted as a measure of the area dA of an element of an m-dimensional subspace spanned by $\dot{x}^i_\alpha$ at the point x^j of X_n in the sense that

$$dA = L(x^i, \dot{x}^i_\alpha)\, dt^i \wedge \ldots \wedge dt^m \tag{2.7}$$

then the tensor (2.6) can be regarded as a suitable areal metric tensor. It is to be noted that $g^{\alpha\beta}_{ij}$ is symmetric in pairs of indices such as (α, i); (β, j). Furthermore, $g^{\alpha\beta}_{ij} \neq g^{\alpha\beta}_{ji}$.

The connection coefficient Γ^i_{kj} defined by (RUND [18])

$$\Gamma^i_{kj} = \frac{1}{m}\left[P^{i\beta}_{kj\beta} + \frac{\partial P^{i\alpha}_{hj\beta}}{\partial \dot{x}^k_\beta}\,\dot{x}^h_\alpha\right] \tag{2.8}$$

where

$$P^{i\beta}_{kj\gamma} = \frac{1}{2}\,G^{hi}_{\alpha\gamma}\left[\frac{\partial g^{\beta\alpha}_{hj}}{\partial x^k} + \frac{\partial g^{\beta\alpha}_{kh}}{\partial x^j} - \frac{\partial g^{\beta\alpha}_{kj}}{\partial x^h} - \frac{\partial^2 g^{\beta\alpha}_{kj}}{\partial \dot{x}^h_\delta\,\partial \dot{x}^l_\lambda}\,f^l_{\delta\lambda}\right] \tag{2.9a}$$

$$G^{\alpha\beta}_{kj} = \frac{1}{2}\,[g^{\alpha\beta}_{kj} + g^{\alpha\beta}_{jk}], \quad G^{\beta\alpha}_{kj}G^{kh}_{\beta\gamma} = \delta^\alpha_\gamma\delta^h_j \tag{2.9b}$$

$$f^l_{\delta\lambda}(x^h, \dot{x}^h_\varepsilon) = \frac{\partial^2 x^l}{\partial t^\delta\,\partial t^\lambda}, \tag{2.9c}$$

has been used to construct the covariant partial derivative of m-linearly independent vector fields $X^i_\varepsilon(x^h, \dot{x}^h_\alpha)$ with respect to x^j and is denoted by $X^i_{\varepsilon|j}$ (RUND [18]). This is given by

$$X^i_{\varepsilon|j} = \frac{\partial X^i_\varepsilon}{\partial x^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{pj}\,\dot{x}^p_\lambda + \Gamma^i_{jl}\,X^l_\varepsilon. \tag{2.10}$$

The connection coefficient Γ^i_{kj} is not in general symmetric in k, j. Therefore, by the process corresponding to which the covariant partial derivative (2.10) has been obtained we can also obtain a second type of covariant partial derivative of X^i_ε with respect to x^j and denote it by $X^i_{\varepsilon\|j}$. Thus

$$X^i_{\varepsilon\|j} = \frac{\partial X^i_\varepsilon}{\partial x^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{jp}\,\dot{x}^p_\lambda + \Gamma^i_{lj}\,X^l_\varepsilon. \tag{2.11}$$

Two curvature tensors have been obtained in [18]. The first curvature tensor is obtained from the integrability conditions associated with the transformation law

$$\bar{\Gamma}^i_{kj} = \frac{\partial \bar{x}^i}{\partial x^p}\left[\frac{\partial^2 x^p}{\partial \bar{x}^k\,\partial \bar{x}^j} + \Gamma^p_{ts}\,\frac{\partial x^t}{\partial \bar{x}^k}\,\frac{\partial x^s}{\partial \bar{x}^j}\right] \tag{2.12}$$

of the connection coefficient. The second curvature tensor is obtained from the commutation rules satisfied by the covariant derivatives (2.10). These curvature tensors are

$$K^i_{khj} = \left(\frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{kh}}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{pj}\,\dot{x}^p_\lambda\right) - \left(\frac{\partial \Gamma^i_{kj}}{\partial x^h} - \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{ph}\,\dot{x}^p_\lambda\right) + \Gamma^i_{pj}\Gamma^p_{kh} - \Gamma^i_{ph}\Gamma^p_{kj} \tag{2.13}$$

and

$$K^{*i}_{khj} = \left(\frac{\partial \Gamma^i_{hk}}{\partial x^j} - \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{pj}\,\dot{x}^p_\lambda\right) - \left(\frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jk}}{\partial \dot{x}^l_\lambda}\,\Gamma^l_{ph}\,\dot{x}^p_\lambda\right) + \Gamma^i_{jp}\Gamma^p_{hk} - \Gamma^i_{hp}\Gamma^p_{jk}, \tag{2.14}$$

respectively.

3. *Lie derivatives of a vector field.* Let us consider an infinitesimal point transformation of the form

$$\bar{x}^i = x^i + v^i(x)\,dt \tag{3.1}$$

where dt is an infinitesimal constant and v^i is a deformation vector of class C^3 defined over the region of X_n. The transformation (3.1) carries the point x^i of the subspace $C_m: x^i = x^i(t^\alpha)$ to the neighbouring point $\bar{x}^i$ of the subspace $\bar{C}_m: \bar{x}^i = \bar{x}^i(t^\alpha)$; t^α being fixed and $v^i(x)=0$ gives the boundary of C_m and $\bar{C}_m$. The corresponding variation of the components $\dot{x}^i_\alpha$ is represented by

$$\dot{\bar{x}}^i_\alpha = \dot{x}^i_\alpha + \frac{\partial v^i}{\partial x^j}\,\dot{x}^j_\alpha\,dt \tag{3.2}$$

where

$$\dot{\bar{x}}^i_\alpha = \frac{\partial \bar{x}^i}{\partial t^\alpha}.$$

Thus the variations of x^i and $\dot{x}^i_\alpha$ under (3.1) is represented in the forms

$$\delta x^i = \bar{x}^i - x^i = v^i(x)\,dt \tag{3.3}$$

$$\delta \dot{x}^i_\alpha = \dot{\bar{x}}^i_\alpha - \dot{x}^i_\alpha = \frac{\partial v^i}{\partial x^j}\,\dot{x}^j_\alpha\,dt. \tag{3.4}$$

Now let us consider a vector field $X^i_\varepsilon(x^h, \dot{x}^h_\alpha)$ defined over X_n. If this vector field is transformed to $X^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha)$ by (3.1) then

$$d^v X^i_\varepsilon = X^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha) - X^i_\varepsilon(x^h, \dot{x}^h_\alpha) = \frac{\partial X^i_\varepsilon}{\partial x^l}\,v^l\,dt + \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\lambda}\,\frac{\partial v^l}{\partial x^j}\,\dot{x}^j_\lambda\,dt. \tag{3.5}$$

On the other hand of we interpret (3.1) as an infinitesimal coordinate transformation, then neglecting higher order terms with respect to dt we have

$$\text{(3.6) (a)}\quad \frac{\partial \bar{x}^i}{\partial x^l} = \delta^i_l + \frac{\partial v^i}{\partial x^l}\,dt \qquad\qquad \text{(b)}\quad \frac{\partial x^l}{\partial \bar{x}^i} = \delta^l_i - \frac{\partial v^l}{\partial x^i}\,dt.$$

When the vector field $X^i_\varepsilon(x^h, \dot{x}^h_\alpha)$ is transformed to $\bar{x}^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha)$ by the coordinate transformation (3.1) then we have

$$d^m X^i_\varepsilon = \bar{X}^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha) - X^i_\varepsilon(x^h, \dot{x}^h_\alpha). \tag{3.7}$$

Since the transformation law for the vector field $X^i_\varepsilon(x^h, \dot{x}^h_\alpha)$ in X_n is

$$X^i_\varepsilon(x^h, \dot{x}^h_\alpha) = \frac{\partial x^i}{\partial \bar{x}^j}\,\bar{X}^j_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha), \tag{3.8}$$

we have from (3.6)b, (3.7) and (3.8)

$$d^m X^i_\varepsilon = X^j_\varepsilon\,\frac{\partial v^i}{\partial x^j}\,dt. \tag{3.9}$$

The Lie derivative of X_ε^l with respect to v^l is defined as (YANO [15], RUND [16])

$$\mathscr{L}_v X_\varepsilon^i = \lim_{dt\to 0} \frac{d^v X_\varepsilon^i - d^m X_\varepsilon^i}{dt}. \tag{3.10}$$

Hence from (3.5), (3.9) and (3.10)

$$\mathscr{L}_v X_\varepsilon^i = \frac{\partial X_\varepsilon^i}{\partial x^l} v^l + \frac{\partial X_\varepsilon^i}{\partial \dot{x}_\lambda^l} \frac{\partial v^l}{\partial x^j} \dot{x}_\lambda^j - X_\varepsilon^j \frac{\partial v^i}{\partial x^j}. \tag{3.11}$$

The Lie derivative of covariant vector field $Y_i^\varepsilon(x^h, \dot{x}_\alpha^h)$ can be obtained in a similar manner. This is given by

$$\mathscr{L}_v Y_i^\varepsilon = \frac{\partial Y_i^\varepsilon}{\partial x^l} v^l + \frac{\partial Y_i^\varepsilon}{\partial \dot{x}_\lambda^l} \frac{\partial v^l}{\partial x^j} \dot{x}_\lambda^j + Y_j^\varepsilon \frac{\partial v^j}{\partial x^i}. \tag{3.12}$$

For a scalar $S(x^h, \dot{x}_\alpha^h)$ we have

$$\mathscr{L}_v S = \frac{\partial S}{\partial x^l} v^l + \frac{\partial S}{\partial \dot{x}_\lambda^l} \frac{\partial v^l}{\partial x^j} \dot{x}_\lambda^j. \tag{3.13}$$

The theorems (3.1) and (3.2) given below are direct consequences of equations (3.11), (3.12), (3.13) and the relation

$$S = X_\varepsilon^i Y_i^\varepsilon.$$

Theorem (3.1). *The Lie derivative of a constant scalar function is zero.*

Theorem (3.2). *The Lie derivative satisfies the Leibnitz rule i.e.*

$$\mathscr{L}_v(X_\varepsilon^i Y_i^\varepsilon) = X_\varepsilon^i(\mathscr{L}_v Y_i^\varepsilon) + (\mathscr{L}_v X_\varepsilon^i) Y_i^\varepsilon. \tag{3.14}$$

The Lie derivative of the partial derivative of X_ε^i with respect to $\dot{x}_\alpha^l$ can be obtained from the first principle and we have

$$\begin{aligned} \mathscr{L}_v \left(\frac{\partial X_\varepsilon^i}{\partial \dot{x}_\alpha^l} \right) &= \frac{\partial^2 X_\varepsilon^i}{\partial x^k \partial \dot{x}_\alpha^l} v^k + \frac{\partial^2 X_\varepsilon^i}{\partial \dot{x}_\beta^k \partial \dot{x}_\alpha^l} \frac{\partial v^k}{\partial x^j} \dot{x}_\beta^j - \\ &- \frac{\partial X_\varepsilon^h}{\partial \dot{x}_\alpha^l} \frac{\partial v^i}{\partial x^h} + \frac{\partial X_\varepsilon^i}{\partial \dot{x}_\alpha^h} \frac{\partial v^h}{\partial x^l} = \frac{\partial}{\partial \dot{x}_\alpha^l} (\mathscr{L}_v X_\varepsilon^i). \end{aligned} \tag{3.15}$$

This proves the following:

Theorem (3.3). *The operations $\mathscr{L}_v$ and $\dfrac{\partial}{\partial \dot{x}_\alpha^l}$ are commutative.*

4. *Lie derivative of a vector field in terms of covariant partial derivatives.* Since the transformation vector v^i depends only on x^i we have from (2.10) and (2.11)

(4.1) (a) $v^i_{|j} = \dfrac{\partial v^i}{\partial x^j} + \Gamma^i_{jl} v^l$ (b) $v^i_{\| j} = \dfrac{\partial v^i}{\partial x^j} + \Gamma^i_{lj} v^l.$

Substituting (2.10), (2.11), (4.1)a and (4.1)b in (3.11) we get (after some simplification)

$$\mathscr{L}_v X^i_\varepsilon = X^i_{\varepsilon|k} v^k + \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\alpha} v^l_{|j} \dot{x}^j_\alpha - X^j_\varepsilon v^i_{\|j} \tag{4.2}$$

or

$$\mathscr{L}_v X^i_\varepsilon = X^i_{\varepsilon\|k} v^k + \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\alpha} v^l_{\|j} \dot{x}^j_\alpha - X^j_\varepsilon v^i_{|j}. \tag{4.3}$$

Generalizing this we may express the Lie derivative of an arbitrary tensor $T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s}$ of X_n in the following two forms

$$\begin{aligned}\mathscr{L}_v T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s} &= v^k T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s|k} + v^1_{|j} \dot{x}^j_\alpha \frac{\partial}{\partial \dot{x}^1_\alpha} [T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s}] - \\ &- \sum_v T^{i_1\dots i_{v-1}k i_{v+1}\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\dots\dots\alpha_r j_1\dots j_s} v^{i_v}_{\|k} + \sum_u T^{i_1\dots i_r\beta_1\dots\dots\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_{u-1}k j_{u+1}\dots j_s} v^k_{\|j_u}.\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mathscr{L}_v T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s} &= v^k T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s\|k} + v^l_{\|j} \dot{x}^j_\alpha \frac{\partial}{\partial \dot{x}^l_\alpha} [T^{i_1\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_s}] - \\ &- \sum_v T^{i_1\dots i_{v-1}k i_{v+1}\dots i_r\beta_1\dots\beta_s}_{\alpha_1\dots\dots\dots\alpha_r j_1\dots j_s} v^{i_v}_{|k} + \sum_u T^{i_1\dots i_r\beta_1\dots\dots\dots\beta_s}_{\alpha_1\dots\alpha_r j_1\dots j_{u-1}k j_{u+1}\dots j_s} v^k_{|j_u}.\end{aligned} \tag{4.5}$$

In particular, the Lie derivative of the metric tensor is given by

$$\mathscr{L}_v g^{\alpha\beta}_{ij} = g^{\alpha\beta}_{ij|k} v^k + 2C^{\alpha\beta\gamma}_{ijk} v^k_{|p} \dot{x}^p_\gamma - g^{\alpha\beta}_{kj} v^k_{\|i} - g^{\alpha\beta}_{ik} v^k_{\|j} \tag{4.6}$$

or

$$\mathscr{L}_v g^{\alpha\beta}_{ij} = g^{\alpha\beta}_{ij\|k} v^k + 2c^{\alpha\beta\gamma}_{ijk} v^k_{\|p} \dot{x}^p_\gamma - g^{\alpha\beta}_{kj} v^k_{|i} - g^{\alpha\beta}_{ik} v^k_{|j} \tag{4.7}$$

where

$$c^{\alpha\beta\gamma}_{ijk} = \frac{1}{2} \frac{\partial g^{\alpha\beta}_{ij}}{\partial \dot{x}^k_\gamma}.$$

From (4.4) and (4.5) it is evident that

Theorem (4.1). *The Lie derivative of a tensor is a tensor of the same order.*

5. *The Lie derivative of connection coefficient.* The Lie derivative of Γ^i_{kj} cannot be found directly from (4.4) or (4.5) as this is not a tensor. We shall, however, evaluate it from the first principle.

By the definition we have

$$d^v \Gamma^i_{kj} = \left(\frac{\partial \Gamma^i_{kj}}{\partial x^l} v^l + \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\alpha} \frac{\partial v^l}{\partial x^p} \dot{x}^p_\alpha \right) dt. \tag{5.1}$$

Substituting (3.6)a and (3.6)b in (2.12) we get (after some simplification)

$$d^m \Gamma^i_{kj} = \bar{\Gamma}^i_{kj}(\bar{x}^h, \dot{\bar{x}}^h_\alpha) - \Gamma^i_{kj}(x^h, \dot{x}^h_\alpha) = -\left[\frac{\partial^2 v^i}{\partial x^k \partial x^j} - \frac{\partial v^i}{\partial x^l} \Gamma^l_{kj} + \frac{\partial v^l}{\partial x^k} \Gamma^i_{lj} + \frac{\partial v^l}{\partial x^j} \Gamma^i_{kl} \right] dt. \tag{5.2}$$

Using the definition of Lie derivative and equations (5.1), (5.2) we have

$$\text{(5.3)}\quad \mathscr{L}_v \Gamma^i_{kj} = \frac{\partial \Gamma^i_{kj}}{\partial x^l} v^l + \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\alpha} \frac{\partial v^l}{\partial x^p} \dot{x}^p_\alpha + \frac{\partial^2 v^i}{\partial x^k \partial x^j} - \frac{\partial v^i}{\partial x^l} \Gamma^l_{kj} + \frac{\partial v^l}{\partial x^k} \Gamma^i_{lj} + \frac{\partial v^l}{\partial x^j} \Gamma^i_{kl}.$$

To express the Lie derivative of Γ^i_{kj} in terms of curvature tensor of X_n we consider expression for $v^i_{|k|j}$ noting that v^i is independent of $\dot{x}^l_\lambda$. Therefore,

$$\text{(5.4)}\quad \begin{aligned} v^i_{|k|j} = v^h \left[\frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{kh}}{\partial \dot{x}^l_\lambda} \Gamma^l_{pj} \dot{x}^p_\lambda + \Gamma^i_{jp} \Gamma^p_{kh} \right] - \Gamma^h_{jk} \frac{\partial v^i}{\partial x^h} - \\ - \Gamma^h_{jk} \Gamma^i_{hl} v^l + \frac{\partial^2 v^i}{\partial x^k \partial x^j} + \Gamma^i_{kh} \frac{\partial v^h}{\partial x^j} + \Gamma^i_{jh} \frac{\partial v^h}{\partial x^k}. \end{aligned}$$

A simple calculation based on the equations (2.13), (5.3) and (5.4) yields

$$\text{(5.5)}\quad \mathscr{L}_v \Gamma^i_{kj} = v^i_{|k|j} + K^i_{kjh} v^h + \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\lambda} v^l_{|p} \dot{x}^p_\lambda + T^h_{jk} v^i_{|h} + T^i_{hj} v^h_{|k}$$

where

$$\text{(5.6)}\quad T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$$

is the torsion tensor associated with the coefficient Γ^i_{jk}. From (5.5) it is evident that

Theorem (5.1). *The Lie derivative of the connection coefficient Γ^i_{kj} is a mixed tensor of third order.*

To find the commutation formulae involving the Lie derivative and covariant partial derivative of the type (2.10) we have from (2.10) and (3.11)

$$\text{(5.7)}\quad (\mathscr{L}_v X^i_\varepsilon)_{|j} = \frac{\partial}{\partial x^j} (\mathscr{L}_v X^i_\varepsilon) + \frac{\partial}{\partial \dot{x}^h_\beta} (\mathscr{L}_v X^i_\varepsilon) \Gamma^h_{pj} \dot{x}^p_\beta + \Gamma^i_{jl} (\mathscr{L}_v X^l_\varepsilon),$$

$$\text{(5.8)}\quad \mathscr{L}_v (X^i_{\varepsilon|j}) = \frac{\partial X^i_{\varepsilon|j}}{\partial x^k} v^k + \frac{\partial X^i_{\varepsilon|j}}{\partial \dot{x}^h_\beta} \frac{\partial v^h}{\partial x^l} \dot{x}^l_\beta - \frac{\partial v^i}{\partial x^l} X^l_{\varepsilon|j} + \frac{\partial v^l}{\partial x^j} X^i_{\varepsilon|l}.$$

Substituting from (3.11) and (2.10) in (5.7) and (5.8) and simplifing we get (after the rearrangement of terms)

$$\text{(5.9)}\quad (\mathscr{L}_v X^i_\varepsilon)_{|j} - \mathscr{L}_v (X^i_{\varepsilon|j}) = -X^l_\varepsilon (\mathscr{L}_v \Gamma^i_{jl}) + \frac{\partial X^i_\varepsilon}{\partial \dot{x}^h_\beta} (\mathscr{L}_v \Gamma^h_{pj}) \dot{x}^p_\beta.$$

If we consider the covariant partial derivative of the type (2.11) then

$$\text{(5.10)}\quad (\mathscr{L}_v X^i_\varepsilon)_{\|j} - \mathscr{L}_v (X^i_{\varepsilon\|j}) = -X^l_\varepsilon (\mathscr{L}_v \Gamma^i_{lj}) + \frac{\partial X^i_\varepsilon}{\partial \dot{x}^h_\beta} (\mathscr{L}_v \Gamma^h_{jp}) \dot{x}^p_\beta.$$

Hence we have

Theorem (5.2). *The operations $\mathscr{L}_v$ and covariant partial derivative are not commutative and the relations* (5.9) *and* (5.10) *hold good.*

Next we consider the covariant partial derivative of $\mathscr{L}_v \Gamma^i_{hk}$ with respect to x^j

$$(\mathscr{L}_v \Gamma^i_{hk})_{|j} = \frac{\partial}{\partial x^j}(\mathscr{L}_v \Gamma^i_{hk}) - \frac{\partial}{\partial \dot{x}^l_\lambda}(\mathscr{L}_v \Gamma^i_{hk})\Gamma^l_{pj}\dot{x}^p_\lambda + (\mathscr{L}_v \Gamma^l_{hk})\Gamma^i_{jl} - (\mathscr{L}_v \Gamma^i_{lk})\Gamma^l_{jh} - (\mathscr{L}_v \Gamma^i_{hl})\Gamma^l_{jk}.$$
(5.11)

Substituting in this expression from (5.3), simplifying and rearranging the terms we obtain

$$\begin{aligned}(5.12)\qquad (\mathscr{L}_v \Gamma^i_{hk})_{|j} &= \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^i_{hk}}{\partial x^j} - \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^a_\lambda}\Gamma^a_{bj}\dot{x}^b_\lambda + \Gamma^i_{ja}\Gamma^a_{hk}\right)v^l + \\ &+ \frac{\partial}{\partial \dot{x}^l_\beta}\left(\frac{\partial \Gamma^i_{hk}}{\partial x^j} - \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^a_\lambda}\Gamma^a_{bj}\dot{x}^b_\lambda + \Gamma^i_{ja}\Gamma^a_{hk}\right)\frac{\partial v^l}{\partial x^p}\dot{x}^p_\beta - \frac{\partial v^i}{\partial x^a}\left(\frac{\partial \Gamma^a_{hk}}{\partial x^j} - \frac{\partial \Gamma^a_{hk}}{\partial \dot{x}^l_\lambda}\Gamma^l_{bj}\dot{x}^b_\lambda + \Gamma^a_{jl}\Gamma^l_{hk}\right) + \\ &+ \frac{\partial v^a}{\partial x^k}\left(\frac{\partial \Gamma^i_{ha}}{\partial x^j} - \frac{\partial \Gamma^i_{ha}}{\partial \dot{x}^l_\lambda}\Gamma^l_{bj}\dot{x}^b_\lambda + \Gamma^i_{jl}\Gamma^l_{ha}\right) + \frac{\partial v^a}{\partial x^h}\left(\frac{\partial \Gamma^i_{ak}}{\partial x^j} - \frac{\partial \Gamma^i_{ak}}{\partial \dot{x}^l_\lambda}\Gamma^l_{bj}\dot{x}^b_\lambda + \Gamma^i_{jl}\Gamma^l_{ak}\right) + \\ &+ \frac{\partial v^a}{\partial x^j}\left(\frac{\partial \Gamma^i_{hk}}{\partial x^a} - \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^l_\lambda}\Gamma^l_{ba}\dot{x}^b_\lambda + \Gamma^i_{al}\Gamma^l_{hk}\right) + \\ &+ \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^l_\lambda}\dot{x}^b_\lambda\left(\frac{\partial \Gamma^l_{bj}}{\partial x^a}v^a + \frac{\partial \Gamma^l_{bj}}{\partial \dot{x}^a_\beta}\frac{\partial v^a}{\partial x^m}\dot{x}^m_\beta + \frac{\partial^2 v^l}{\partial x^b \partial x^j} - \frac{\partial v^l}{\partial x^a}\Gamma^a_{bj} + \frac{\partial v^a}{\partial x^b}\Gamma^l_{aj} + \frac{\partial v^a}{\partial x^j}\Gamma^l_{ba}\right) - \\ &- \Gamma^l_{jh}(\mathscr{L}_v \Gamma^i_{lk}) + \textit{terms which are symmetric in } j \textit{ and } h.\end{aligned}$$

In this relation we interchange the indices j, h and subtract the result from (5.12). This process gives the following commutation formulae.

(5.13)
$$(\mathscr{L}_v \Gamma^i_{hk})_{|j} - (\mathscr{L}_v \Gamma^i_{jk})_{|h} = \mathscr{L}_v K^{*i}_{khj} + T^l_{hj}(\mathscr{L}_v \Gamma^i_{lk}) + \frac{\partial \Gamma^i_{hk}}{\partial \dot{x}^l_\lambda}(\mathscr{L}_v \Gamma^l_{bj})\dot{x}^b_\lambda - \frac{\partial \Gamma^i_{jk}}{\partial \dot{x}^l_\lambda}(\mathscr{L}_v \Gamma^l_{bh})\dot{x}^b_\lambda$$

where we have used the relation (2.14), (5.6) and (5.3).

In a similar way we consider the covariant partial derivative (of $\mathscr{L}_v \Gamma^i_{hk}$) of the type (2.11) and obtain the following commutation rule

(5.14)
$$(\mathscr{L}_v \Gamma^i_{kh})_{\|j} - (\mathscr{L}_v \Gamma^i_{kj})_{\|h} = \mathscr{L}_v R^i_{khj} + T^l_{jh}(\mathscr{L}_v \Gamma^i_{kl}) + \frac{\partial \Gamma^i_{kh}}{\partial \dot{x}^l_\lambda}(\mathscr{L}_v \Gamma^l_{jb})\dot{x}^b_\lambda - \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\lambda}(\mathscr{L}_v \Gamma^l_{hb})\dot{x}^b_\lambda$$

where

$$(5.15)\quad R^i_{khj} = \left(\frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{kh}}{\partial \dot{x}^l_\lambda}\Gamma^l_{jp}\dot{x}^p_\lambda\right) - \left(\frac{\partial \Gamma^i_{kj}}{\partial x^h} - \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\lambda}\Gamma^l_{hp}\dot{x}^p_\lambda\right) + \Gamma^i_{lj}\Gamma^l_{kh} - \Gamma^i_{lh}\Gamma^l_{kj}.$$

It can be easily verified that for $m=1$ the space X_n is a Finsler space and the torsion tensor T^i_{kj} vanishes. In this case the connection coefficient Γ^i_{kj} given by (2.8) reduces to Berwald's connection ciefficient G^i_{kj} (RUND [16] page 79), the curvature tensors K^i_{khj}, K^{*i}_{khj} and R^i_{khj} given by (2.13), (2.14) and (5.15) reduce to Berwald's curvature tensor H^i_{khj} (RUND [16] page 125) and each of the covariant partial derivatives (2.10), (2.11) will reduce to the Berwald's covariant partial derivative (RUND [16] page 80). The expression (5.5) in this particular case will take the form

$$\mathscr{L}_v G^i_{kj} = v^i_{(k)(j)} + H^i_{kjh} v^h + \frac{\partial G^i_{kj}}{\partial \dot{x}^l} v^l_p \dot{x}^p \tag{5.16}$$

which is the Lie derivative of Berwald's connection coefficient already obtained (RUND [16] page 220).

Also in this particular case the expressions (5.13) and (5.14) will take the form

$$(\mathscr{L}_v G^i_{hk})_{(j)} - (\mathscr{L}_v G^i_{jk})_{(h)} = \mathscr{L}_v H^i_{khj} + \frac{\partial G^i_{hk}}{\partial \dot{x}^l} (\mathscr{L}_v G^l_{bj}) \dot{x}^b - \frac{\partial G^i_{jk}}{\partial \dot{x}^l} (\mathscr{L}_v G^l_{bh}) \dot{x}^b. \tag{5.17}$$

which has already been obtained (Yano [15] page 189, Prasad [20], [21]).

6. *Areal motion.* In this section we introduce the concept of an areal motion. When the fundamental metric funtion $L(x^h, \dot{x}^h_\alpha)$ satisfies the relation

$$\mathscr{L}_v L(x^h, \dot{x}^h_\alpha) = 0 \tag{6.1}$$

the transformation (3.1) does not change the area

$$A = \int_{(m)} L(x^h, \dot{x}^h_\alpha)\, dt^1 \wedge \ldots \wedge dt^m$$

of an m-dimensional subspace. On account of this reason we give the following definition:

Definition (6.1): The transformation given by (3.1) is called an areal motion if $\mathscr{L}_v L(x^h, \dot{x}^h_\alpha) = 0$.

Theorem (6.1). *In order that the space admits an areal motion it is necessary and sufficient that the Lie derivative of the metric tensor* $g^{\alpha\beta}_{ij}$ *vanishes.*

PROOF. The necessary part follows from theorem (3.3) and the equations (6.1), (2.6).

The sufficient part follows from theorems (3.1), (3.2), the equation (2.5) and the fact that $\mathscr{L}_v \dot{x}^i_\alpha = 0$.

Applying the formulae (5.9) and (5.10) to the areal metric tensor $g^{\alpha\beta}_{ij}$ we have

$$(\mathscr{L}_v g^{\alpha\beta}_{ij})_{|k} - \mathscr{L}_v (g^{\alpha\beta}_{ij|k}) = g^{\alpha\beta}_{lj} (\mathscr{L}_v \Gamma^l_{ki}) + g^{\alpha\beta}_{il} (\mathscr{L}_v \Gamma^l_{kj}) + 2c^{\alpha\beta\gamma}_{ijh} (\mathscr{L}_v \Gamma^h_{pk}) \dot{x}^p_\gamma \tag{6.2}$$

$$(\mathscr{L}_v g^{\alpha\beta}_{ij})_{\|k} - \mathscr{L}_v (g^{\alpha\beta}_{ij\|k}) = g^{\alpha\beta}_{lj} (\mathscr{L}_v \Gamma^l_{ik}) + g^{\alpha\beta}_{il} (\mathscr{L}_v \Gamma^l_{jk}) + 2c^{\alpha\beta\gamma}_{ijh} (\mathscr{L}_v \Gamma^h_{kp}) \dot{x}^p_\gamma. \tag{6.3}$$

The expressions (6.2) and (6.3) are too complicated to make further study possible. We, therefore, assume that the connection coefficient Γ^i_{kj} and the metric tensor $g^{\alpha\beta}_{ij}$ satisfies the conditions

$$T^i_{kj} = 0 \tag{6.4a}$$

$$\mathscr{L}_v (g^{\alpha\beta}_{ij|k}) = 0 \tag{6.4b}$$

$$|M^h_l| = \det \| \delta^h_i g^{\alpha\beta}_{lj} + \delta^h_j g^{\alpha\beta}_{il} + 2c^{\alpha\beta\gamma}_{ijl} \dot{x}^h_\gamma \| \neq 0. \tag{6.4c}$$

It should be noted that there cannot in general exist a symmetric connection for which the covariant derivative of the metric tensor vanish (RUND [18]). However, we can choose the vector field $v^i(x)$ such that $\mathscr{L}_v(g^{\alpha\beta}_{ij|k})=0$.

In order to discuss the nature of the areal motion we give the following definition (PRASAD [20], [21]).

Definition (6.2): The transformation given by (3.1) is called an affine motion if $\mathscr{L}_v\Gamma^i_{kj}=0$.

Definition (6.3): The areal space X_n is said to admit a curvature collineation if there exist a vector $v^i(x)$ satisfying the condition

$$\mathscr{L}_v K^i_{khj} = 0 \tag{6.5}$$

The condition (6.4)a implies that the covariant partial derivative (2.10) and (2.11) are equivalent and

$$K^i_{khj} = K^{*i}_{khj} = R^i_{khj}. \tag{6.6}$$

Hence under the conditions (6.4)a and (6.4)b the equations (6.2) and (6.3) will reduce to

$$(\mathscr{L}_v g^{\alpha\beta}_{ij})_{|k} = (\delta^h_i g^{\alpha\beta}_{lj} + \delta^h_j g^{\alpha\beta}_{il} + 2c^{\alpha\beta\gamma}_{ijl}\dot{x}^h_\gamma)(\mathscr{L}_v\Gamma^l_{kh}). \tag{6.7}$$

The theorem (6.1) and relation (6.7) yield

Theorem (6.2). *Under the conditions* (6.4)a, (6.4)b *and* (6.4)c *every areal motion is affine motion.*

From equations (5.13) and (6.6) we have

Theorem (6.3). *Every affine motion is a curvature collineation.*

From theorems (6.2) and (6.3) we obtain

Theorem (6.4). *Every areal motion is a curvature collineation under the conditions* (6.4)a, (6.4)b *and* (6.4)c.

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References

[1] M. S. KNEBLMAN, Collineations and motions in generalized space. *American J. Math.* **51** (1929), 527—564.
[2] E. T. DAVIES, Motion in a metric space based on the notion of area. *Quart. J. Math.* **16** (1945), 22—30.
[3] E. T. DAVIES, The geometry of the multiple integral. *J. London Math. Soc.* **20** (1945), 163—170.
[4] E. T. DAVIES, Areal spaces. *Ann. Mat. Pura Appl.* IV, *Ser.* **55** (1961), 63—76.
[5] E. T. DAVIES, On the use of osculating spaces. *Tensor N. S.* **14** (1963), 86—98.
[6] A. KAWAGUCHI, On areal spaces I, II, III. *Tensor N. S.* **1** (1951), 14—45, 67—88, 89—103.
[7] A. KAWAGUCHI, Theory of areal spaces. *Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl.* (5) **12** (1953), 373—386.

[8] A. KAWAGUCHI and Y. KATSURADA, On areal spaces IV. *Tensor N. S.* **1** (1951), 137—156.
[9] A. KAWAGUCHI and Y. KATSURADA, On a connection in an areal space. *Jap. J. Math.* **21** (1951), 249—262.
[10] A. KAWAGUCHI and K. TANDAI, On areal spaces V. *Tensor N. S.* **2** (1952), 47—58.
[11] K. TANDAI, On areal spaces VI. *Tensor N. S.* **3** (1953), 40—45.
[12] K. TANDAI, On areal spaces VII. *Tensor N. S.* **4** (1954), 78—90.
[13] K. TANDAI, On areal spaces VIII. *Tensor N. S.* **10** (1960), 161—166.
[14] K. TANDAI, On general connections in an areal space I. *Tensor N. S.* **13** (1963), 277—291.
[15] K. YANO, The theory of Lie derivatives and its applications. *Amsterdam.* 1957.
[16] H. RUND, The differential geometry of Finsler spaces. *Springer Verlag.* 1959.
[17] H. RUND, The Hamilton—Jacobi theory in the calculus of variations. *London and New York.* 1966.
[18] H. RUND, A geometrical theory of multiple integral problems in the calculus of variations. *Can. J. Math.* **20** (1968), 639—657.
[19] T. IGARASHI, On Lie derivatives on areal spaces. *Tensor N. S.* **18** (1967), 205—211.
[20] B. N. PRASAD, Curvature collineations in Finsler space. *Atti della Accad. Naz. Lincei Rendiconti.* **49** (1970), 194—198.
[21] B. N. PRAFAD, Special curvature collineations in Finsler space. *Atti Accad. Naz. Lincei Rendiconti.* **50** (1971), 82—87.

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