A sine functional equation in Banach algebras

By B. NAGY (Budapest)

The functional equation $f(x+y+\alpha)-f(x-y+\alpha)=2f(x)f(y)$, where $\alpha\in R$ (the field of real numbers) was essentially first considered by E. B. VAN VLECK (cf. [1], [11]) for the case when $f: R\to R$ is continuous. The aim of this paper is to investigate this equation in the cases when $f: R\to B$, where B is a complex algebra, or Banach algebra, or the algebra of the bounded linear operators in a complex Banach space X. In the two latter cases measurability conditions will be imposed, which also ensure continuity of the solutions.

If B is a complex algebra, then the general solution is given in terms of its odd component, which on the other hand, is in connection with a type of exponential function. In the case of a complex Banach algebra B the concept of the generator element is defined and characterized, and the strongly measurable solutions are given in terms of the generator.

If $B=B(X)$ is the algebra of bounded linear operators in a complex Banach space X, then essentially continuity of the solutions in the strong operator topology is assumed and similar problems are investigated. Theorem 8 deals with the case when $X=H$ is a Hilbert space. Note that sometimes we use the terminology of [6] without explicitly mentioning this.

1. The equation in algebras and Banach algebras

Definition. Let R be the real field, B an algebra over the complex field, $S: R\to B$ a map, $\alpha\in R$. We shall say that $S\in V(\alpha)$, if for every $(\xi,\eta)\in R\times R$ we have

$$S(\xi+\eta+\alpha)-S(\xi-\eta+\alpha) = 2S(\xi)S(\eta) \tag{1}$$

We say that $S\in V_0(\alpha)$ if S is odd and satisfies (1).

Fundamental is in what follows the following

Theorem 1. *If $S\in V_0(\alpha)$, then $S(-\alpha)=j$ is idempotent, and $jS(\xi)=S(\xi)j=S(\xi)$ holds for every $\xi\in R$. Putting $C(\xi)=S(\xi-\alpha)$, $C(\xi)$ satisfies the equation $C(\xi+\eta)+C(\xi-\eta)=2C(\xi)C(\eta)$ (d'Alembert's equation), and $E(\xi)=S(\xi-\alpha)+i\,S(\xi)$ satisfies the equations*

$$\begin{aligned} E(\xi+\eta) &= E(\xi)E(\eta) \\ E(\xi+\alpha) &= -iE(\xi) \end{aligned} \tag{2}$$

Conversely, if $E: R\to B$ satisfies (2), *then* $S(\xi)=\frac{1}{2i}\{E(\xi)-E(-\xi)\}\in V_0(\alpha)$.

PROOF. 1. S is odd, therefore $S(0)=0$. Setting $\xi=0$ in (1), we have

$$S(\eta+\alpha) = S(-\eta+\alpha) \tag{3}$$

and hence

$$S(\eta)S(\xi) = \frac{1}{2}\{S(\xi+\eta+\alpha)-S(\eta-\xi+\alpha)\} = S(\xi)S(\eta) \quad \text{for} \quad \xi,\eta\in R.$$

Since S is odd, putting $\xi=-\alpha$ in (1), we obtain that $S(\eta)=S(-\alpha)S(\eta)$ $(\eta\in R)$ and $S(-\alpha)=j$ is idempotent. For the same reason (3) yields

$$S(\xi+\alpha) = -S(\xi-\alpha) \quad (\xi\in R) \tag{4}$$

and, considering (1) for $(\xi-\alpha, \eta-\alpha)\in R\times R$, we conclude that

$$S(\xi+\eta-\alpha)+S(\xi-\eta-\alpha) = 2S(\xi-\alpha)S(\eta-\alpha) \tag{5}$$

that is $C(\xi)=S(\xi-\alpha)$ satisfies d'Alembert's equation. In view of the definition of E

$$E(\xi)E(\eta) = C(\xi)C(\eta)-S(\xi)S(\eta)+i\{S(\xi)C(\eta)+C(\xi)S(\eta)\}.$$

Here we have $C(\xi)C(\eta)-S(\xi)S(\eta)=C(\xi)C(\eta)-C(\xi+\alpha)C(\eta+\alpha)=1/2\{C(\xi+\eta)+$ $+C(\xi-\eta)-C(\xi+\eta+2\alpha)-C(\xi-\eta)\}=C(\xi+\eta)$ according to (4) and (5). Similarly, by (1) and (4) we get

$$S(\xi)C(\eta)+C(\xi)S(\eta) = S(\xi)S(\eta-\alpha)+S(\xi-\alpha)S(\eta) =$$

$$= \frac{1}{2}\{S(\xi+\eta)-S(\xi-\eta+2\alpha)+S(\xi+\eta)-S(\xi-\eta)\} = S(\xi+\eta)$$

and thus $E(\xi)E(\eta)=E(\xi+\eta)$. Moreover, by (4)

$$E(\xi+\alpha) = C(\xi+\alpha)+iS(\xi+\alpha) = S(\xi)-iC(\xi) = -iE(\xi).$$

2. To prove the converse we observe that $E(\xi-\alpha)=iE(\xi)$ and

$$S(\xi+\eta+\alpha)-S(\xi-\eta+\alpha) =$$

$$= \frac{1}{2i}\{E(\xi+\eta+\alpha)-E(-\xi-\eta-\alpha)-E(\xi-\eta+\alpha)+E(-\xi+\eta-\alpha)\} =$$

$$= \frac{1}{2i}\{-iE(\xi+\eta)-iE(-\xi-\eta)+iE(\xi-\eta)+iE(-\xi+\eta)\} = 2S(\xi)S(\eta).$$

Since S obviously is odd, $S\in V_0(\alpha)$, and the proof is complete.

Definition. The functions C, E occurring in Theorem 1 will be called the functions C, E associated with $S\in V_0(\alpha)$.

Remark. If B_u is a subalgebra of B with unit u, $S: R\to B_u$, $S\in V(\alpha)$, and for some $\xi_0\in R$ there exists the inverse of $S(\xi_0)$ in B_u (denoted by $S(\xi_0)_u^{-1}$), then $S\in V_0(\alpha)$. Indeed, by (1) we have for every $\eta\in R$

$$S(\xi_0)S(-\eta) = -S(\xi_0)S(\eta)$$

and, multiplying by $S(\xi_0)_u^{-1}$, we obtain that S is odd.

Theorem 2. *If* $S\in V(\alpha)$, $S_e(\xi)=1/2\{S(\xi)+S(-\xi)\}$, $S_0(\xi)=1/2\{S(\xi)-S(-\xi)\}$, *then* $S_0\in V_0(\alpha)$ *and* $S_e(\xi)=a\ S_0(\xi+\alpha)+a-aj$, *where* $a=S_e(2\alpha)$, $j=S_0(-\alpha)$ *is idempotent, and* $a^2=0$, $ja=0$. *Conversely, if* $S_0\in V_0(\alpha)$, $j=S_0(-\alpha)$ *and there exists an* $a\in B$ *such that* $a^2=0$, $ja=0$, *then* $S(\xi)$ *defined by* $S_0(\xi)+aS_0(\xi+\alpha)+$ $+a-aj$ *belongs to* $V(\alpha)$ *and* S_0 *is the odd part of* S.

PROOF. 1. If $S\in V(\alpha)$, then $S(\xi)S(-\eta)=-S(\xi)S(\eta)$ for $\xi,\eta\in R$, and hence

$$S(\xi)S_e(\eta)=0\quad(\xi,\eta\in R). \tag{6}$$

Putting the pair $(-\xi,\eta)$ in (6) we get

$$S_e(\xi)S_e(\eta)=0 \tag{6a}$$
$$S_0(\xi)S_e(\eta)=0 \qquad (\xi,\eta\in R) \tag{6b}$$

With $\xi=-\alpha$ in (1) we obtain

$$S(-\alpha)S(\eta)=\frac{1}{2}\{S(\eta)-S(-\eta)\}=S_0(\eta) \tag{7}$$

while (1) becomes because of (6)

$$S(\xi+\eta+\alpha)-S(\xi-\eta+\alpha)=2S(\xi)S_0(\eta) \tag{8}$$

Multiplying (8) by $S(-\alpha)$ from the left, we have by (7)

$$S_0(\xi+\eta+\alpha)-S_0(\xi-\eta+\alpha)=2S_0(\xi)S_0(\eta) \tag{9}$$

and thus $S_0\in V_0(\alpha)$. Using (8) again, we obtain

$$S_e(\xi+\eta+\alpha)-S_e(\xi-\eta+\alpha)=2S_e(\xi)S_0(\eta) \tag{10}$$

Considering (10) for the pair $(\xi-\alpha,\eta)$ implies

$$S_e(\xi-\alpha)S_0(\eta)=\frac{1}{2}\{S_e(\xi+\eta)-S_e(\xi-\eta)\}=S_e(\eta-\alpha)S_0(\xi)\quad(\xi,\eta\in R) \tag{11}$$

for the middle part is symmetric. From (9) it follows that $S_0(-\alpha)=j$ is idempotent, and (11) for the pair $(\xi+\alpha,-\alpha)$ gives that

$$S_e(\xi)\cdot j=S_e(2\alpha)S_0(\xi+\alpha). \tag{12}$$

Then for every $\xi,\eta\in R$ if follows that

$$S_e(\xi)S_0(\eta)=S_e(\xi)\cdot j\cdot S_0(\eta)=S_e(2\alpha)S_0(\xi+\alpha)S_0(\eta). \tag{13}$$

Putting $\eta=\xi-\alpha$ in (10) and making use of (13) gives

$$S_e(2\xi)=S_e(2\alpha)+2S_e(2\alpha)S_0(\xi+\alpha)S_0(\xi-\alpha)$$

and by (9) we get

$$S_e(2\xi)=S_e(2\alpha)+S_e(2\alpha)\{S_0(2\xi+\alpha)-S_0(3\alpha)\}.$$

From (4) it can be seen that S_0 is periodic with period $4|\alpha|$, thus with the notation of the theorem we have $S_e(\xi)=a+a\{S_0(\xi+\alpha)-j\}$. Moreover, from (6a) and (6b) we see that $a^2=0$, $ja=0$.

2. If $S_0 \in V_0(\alpha)$ and $S(\xi)$ is of the asserted form, then with the notation $S_e(\xi) = S(\xi) - S_0(\xi)$ we obtain that $S_0(\xi) S_e(\eta) = 0$ $(\xi, \eta \in R)$, for $ja = 0$. Similarly, since $a^2 = 0$, it follows that $S_e(\xi) \cdot S_e(\eta) = \{a + a[S_0(\xi+\alpha) j - j]\} \cdot \{a + a[S_0(\eta+\alpha) - j]\} = 0$. Thus, using (9), we establish that $2S(\xi) S(\eta) = 2\{a + a[S_0(\xi+\alpha) - j] + S_0(\xi)\} S_0(\eta) = 2 \cdot S_0(\xi) S_0(\eta) + 2aS_0(\xi+\alpha) S_0(\eta) + 2aS_0(\eta) - 2ajS_0(\eta) = S_0(\xi+\eta+\alpha) - S_0(\xi-\eta+\alpha) + a\{S_0(\xi+\eta+2\alpha) - S_0(\xi-\eta+2\alpha)\} = S(\xi+\eta+\alpha) - S(\xi-\eta+\alpha)$, hence $S \in V(\alpha)$. The last statement is obvious, because in view of $S_0 \in V_0(\alpha)$ $S_0(-\xi+\alpha) = S_0(\xi+\alpha)$ holds.

Corollary 1. If $S \in V(\alpha)$ and $S(0) = 0$, then $S \in V_0(\alpha)$.

PROOF. Using the notations of the preceding theorem $S_0 \in V_0(\alpha)$, and (4) implies that $S_0(\alpha) = -j$. By Theorem 2, $0 = S(0) = a - 2aj$, thus $aS_0(\xi) = 2ajS_0(\xi) = 2aS_0(\xi)$, hence $aS_0(\xi) = 0$ for $\xi \in R$. Therefore $S_e(\xi) = a - aj = a/2$ for $\xi \in R$. However, $S_e(0) = 0$ by assumption, thus $S_e(\xi) = 0$ for $\xi \in R$, and $S = S_0 \in V_0(\alpha)$.

Corollary 2. $S \in V(0)$ implies $S(\xi) = a$ for $\xi \in R$, where $a^2 = 0$. Conversely, if $S(\xi) = a$ for $\xi \in R$ with $a^2 = 0$, then $S \in V(0)$.

PROOF. If $S \in V(0)$, then Theorem 2 and (4) imply $S_0(\xi) = -S_0(\xi)$, $S_0(\xi) = 0$ for $\xi \in R$. Hence $j = 0$, and $S(\xi) = a$ with $a^2 = 0$. The converse is trivial.

In view of this corollary we shall always assume that $\alpha \neq 0$.

Example. Let X be a Banach space, $B = B(X)$ the Banach algebra of the bounded linear operators in X, $J \in B$ an idempotent (projection) operator, $J \neq 0$, $J \neq I$. Then there exists an $A \in B$ such that $A^2 = 0$, $JA = 0$, $AJ \neq 0$.

Indeed, denoting the range of $T \in B$ by $R(T)$, we have $X = R(J) \oplus R(I-J)$, and there exist elements $u \in R(I-J)$, $u \neq 0$ and $u^* \in X^*$ such that $u^*(u) = 0$, and $u^*\{R(J)\} \not\equiv 0$. For $x \in X$ set $Ax = u^*(x) \cdot u$, then $A \in B$, $R(A) \subset R(I-J)$, thus $JA = 0$. On the other hand, $AJx = u^*(Jx) \cdot u \not\equiv 0$, and $A^2 x = A\{u^*(x) \cdot u\} = u^*(x) \cdot u^*(u) \cdot u = 0$ for every $x \in X$, thus A fulfils the stated requirements.

Definition. We shall say that $S \in V(\alpha, B, j, a)$ if $S \in V(\alpha)$, B is a complex Banach algebra, $S_0(-\alpha) = j$, $S_e(2\alpha) = a$ and S is strongly measurable. We shall omit any of the variables of V if it is not emphasized.

Theorem 3. *If $S \in V(\alpha, B, j, .)$, then there exist one and only one $g_0 \in jBj$ and exactly one $a \in B$ such that $S(\xi) = \sin(\xi g_0) - a \cos(\xi g_0) + a - aj$, where by definition*

$$\sin(\xi g_0) = \sum_{n=0}^{\infty} (-1)^n \frac{(\xi g_0)^{2n+1}}{(2n+1)!}, \quad \cos(\xi g_0) = j + \sum_{n=1}^{\infty} (-1)^n \frac{(\xi g_0)^{2n}}{(2n)!}.$$

Here $a = S_e(2\alpha)$, $a^2 = 0$, $ja = 0$, S is strongly differentiable at each point $\xi \in R$ and $S'(\xi) = g_0 \cos(\xi g_0) + ag_0 \sin(\xi g_0)$. Moreover, $S'(0) = S_0'(0) = g_0$.

PROOF. Under the conditions of the theorem $S_0 \in V(\alpha, B, j, .)$ and, by Theorem 1, the function E associated with S_0 is also strongly measurable, and $E(0) = j$. Then by [6], 9.4. there exists exactly one $g \in jBj$ such that

$$E(\xi) = j + \sum_{n=1}^{\infty} \frac{(\xi g)^n}{n!} = \exp(\xi g),$$

and this series converges absolutely for $\xi \in R$. Writing $g_0 = -ig \in jBj$, we obtain that $S_0(\xi) = \sin(\xi g_0)$, and because of $S_0'(0) = g_0$ the uniqueness of g_0 is proved. Taking into account (4) and Theorem 1, we get $S_0(\xi+\alpha) = -S_0(\xi-\alpha) = -C(\xi)$, where C is the function associated with S_0 according to Theorem 1. But then

$$S_0(\xi+\alpha) = -\frac{1}{2}\{E(\xi)+E(-\xi)\} = -\frac{1}{2}\{\exp(i\xi g_0)+\exp(-i\xi g_0)\} = -\cos(\xi g_0),$$

and the remaining assertions are contained in Theorem 2.

Corollary (cf. [7], [2]). If B is a complex Banach-algebra, $C: R \to B$ is a strongly measurable function satisfying $C(\xi+\eta)+C(\xi-\eta) = 2C(\xi)C(\eta)$, and if for some $\alpha \in R$ $(\alpha \neq 0)$ $C(\alpha) = 0$ holds, then $C(0) = j$ is idempotent and there exists a (not unique) $b \in jBj$ such that $C(\xi) = \cos(\xi b) = j + \sum_{n=1}^{\infty} (-1)^n \frac{(\xi b)^{2n}}{(2n)!}$.

PROOF. From the assumptions we obtain that $C(\xi+\alpha) = -C(\xi-\alpha)$ for $\xi \in R$. Putting the pair $(\xi+\alpha, \eta+\alpha)$ in d'Alembert's equation, the preceding relation gives

$$C(\xi+\eta+2\alpha) - C(\xi-\eta+2\alpha) = 2C(\xi+\alpha)C(\eta+\alpha). \tag{14}$$

If we set $S(\xi) = C(\xi+\alpha)$, then $S(0) = 0$ and (14) imply that $S \in V_0(\alpha)$, and clearly, S is strongly measurable. Thus, as in Theorem 3, for the function E associated with S we get $E(\xi) = j + \sum_{n=1}^{\infty} \frac{(\xi b)^n}{n!}$, where $j = C(0) = S(-\alpha)$ is idempotent and $b \in jBj$. From this it follows that $C(\xi) = 1/2\{E(\xi)+E(-\xi)\} = \cos(\xi b)$, while $C(\xi) = \cos\{\xi(-b)\}$ is a trivial different representation.

Definition. If $S \in V(\alpha, B, ., .)$, then $\frac{dS}{d\xi}(0)$ will be called the generator element of S.

A characterization of the genarator elements is given in the following

Theorem 4. *Let B be a complex Banach algebra, $j \in B$ idempotent, $\alpha \in R$. $g_0 \in jBj$ is the generator element of some $S \in V(\alpha, B, j, .)$ if and only if $g_0 = \sum_{\beta=1}^{n} \lambda_\beta j_\beta$, where $\lambda_\beta = \frac{\pi}{2\alpha}(4k_\beta - 1)$ (k_β are different integers for $\beta = 1, 2, \ldots, n$), $j_\beta \in jBj$ are idempotents,*

$$\sum_{\beta=1}^{n} j_\beta = j, \quad j_\beta j_\gamma = \delta_{\beta\gamma} j_\beta$$

$(\gamma = 1, 2, \ldots, n)$. *In this case we have*

$$S_0(\xi) = \sum_{\beta=1}^{n} j_\beta \cdot \sin(\lambda_\beta \cdot \xi), \quad S_0(\xi+\alpha) = -\sum_{\beta=1}^{n} j_\beta \cdot \cos(\lambda_\beta \cdot \xi).$$

PROOF. According to the preceding theorems $g_0 \in jBj$ is the generator element of some $S \in V(\alpha; B, j, .)$ if and only if $E(\xi) = S_0(\xi - \alpha) + iS_0(\xi) = \exp(i\xi g_0)$, where exp denotes the exponential function in jBj, and $E(\alpha) = \exp(i\alpha g_0) = -ij$. The latter is the case exactly when $i\alpha g_0 = \log(-ij)$, where log denotes logarithm in jBj. A value of this is clearly $\log^*(-ij) = -i\frac{\pi}{2} j$, and the spectrum of $-i\frac{\pi}{2} \cdot j \in jBj$ is $\left\{-i\frac{\pi}{2}\right\}$, which is incongruent $(\mathrm{mod}\, 2\pi i)$ (see [5], p. 54.). Hence every logarithm of $-ij$ is of the following form (see loc. cit.) : $\log(-i \cdot j) = -i\frac{\pi}{2} \cdot j + i2\pi \sum_{\beta=1}^{n} k_\beta \cdot j_\beta$, where k_β $(\beta = 1, 2, \ldots, n)$ are different integers, $j_\beta \in jBj$, $j_\beta \cdot j_\gamma = \delta_{\beta\gamma} \cdot j_\beta$ $(\beta, \gamma = 1, 2, \ldots, n)$, $\sum_{\beta=1}^{n} j_\beta = j$. Therefore we obtain that

$$g_0 = \frac{\pi}{2\alpha}\left\{-\sum_{\beta=1}^{n} j_\beta + 4\sum_{\beta=1}^{n} k_\beta j_\beta\right\} = \frac{\pi}{2\alpha}\sum_{\beta=1}^{n} (4k_\beta - 1) j_\beta.$$

Because of the properties of the elements j, j_β we have in this case

$$E(\xi) = \exp\left(i\xi \sum_{\beta=1}^{n} \lambda_\beta j_\beta\right) = \prod_{\beta=1}^{n} \exp(i \cdot \xi \cdot \lambda_\beta \cdot j_\beta) = \prod_{\beta=1}^{n} \{j + j_\beta \cdot [\exp(i\xi\lambda_\beta) - 1]\} =$$

$$= j + \sum_{\beta=1}^{n} j_\beta[\exp(i\xi\lambda_\beta) - 1] = \sum_{\beta=1}^{n} j_\beta \exp(i\xi\lambda_\beta),$$

and hence $S_0(\xi) = \sum_{\beta=1}^{n} j_\beta \sin(\lambda_\beta \cdot \xi)$. For the function C associated with S_0 $S_0(\xi + \alpha) = -S_0(\xi - \alpha) = -C(\xi)$ holds and thus the remaining statement is also proved.

2. The equation in the algebra $B(X)$.

Let X be a complex Banach space and $B(X)$ the algebra of the bounded linear operators in X.

Definition. We say that $S \in V(\alpha, B(X), J, A)$ if $S: R \to B(X)$ satisfies (1), $S_0(-\alpha) = J$ is a projection operator, $S_e(2\alpha) = A$ and S is a strongly measurable operator function. Moreover, we say that $S \in V_0(\alpha, B(X), J)$ if, in addition, S is odd (then necessarily $A = 0$).

Clearly, if $S \in V(\alpha, B(X), J, A)$, then S_0 and the function E associated with S_0 are also strongly measurable, hence E and S are continuous in the strong operator topology of $B(X)$.

Definition. If $S \in V(\alpha, B(X), J, A)$, then we set $S'(\xi)x = \lim_{\eta \to 0} \frac{S(\xi + \eta) - S(\xi)}{\eta} x$ if this limit exists in the strong topology of X, and then we write $x \in D\{S'(\xi)\}$. We denote $S'(0)$ by $G_0 = G_0(S)$, and call it the genereator operator of S.

Lemma 1. *If* $S\in V_0(\alpha, B(X), J)$ *and* $x\in D(G_0)$, *then for every* $\xi\in R$ *we have* $x\in D\{S'(\xi)\}$, $S(\xi)x\in D(G_0)$ *and*

$$S'(\xi)x = S(\xi-\alpha)G_0x = G_0S(\xi-\alpha)x.$$

PROOF. By (1) and the strong continuity of S, we obtain for $x\in D(G_0)$, $\xi\in R$ that there exists

$$(15)\qquad S'(\xi)x = \lim_{\eta\to 0}\frac{S(\xi+\eta)-S(\xi)}{\eta}x = \lim_{\eta\to 0}\frac{S(\xi-\alpha+\eta/2)S(\eta/2)}{\eta/2}x = S(\xi-\alpha)G_0x.$$

On the other hand, for $x\in D(G_0)$

$$G_0S(\xi-\alpha)x = \lim_{\eta\to 0}\frac{1}{\eta}S(\eta)S(\xi-\alpha)x = \lim_{\eta\to 0}\frac{S(\xi+\eta)-S(\xi-\eta)}{2\eta}x =$$

$$= \lim_{\eta\to 0}\frac{1}{2}\cdot\frac{S(\xi+\eta)-S(\xi)-\{S(\xi-\eta)-S(\xi)\}}{\eta}x = S'(\xi)x$$

exists because of (15), and the proof is complete.

As a corollary we get that $x\in D(G_0)$ implies $G_0x=S(-\alpha)G_0x$, hence $R(G_0)\subset\subset R\{S(-\alpha)\}=R(J)$.

Lemma 2. *If* $S\in V(\alpha, B(X), J, A)$ *and* S_0 *is its odd part, then* $G_0(S)=G_0(S_0)$.

PROOF. If $x\in D\{G_0(S)\}$, then

$$G_0(S_0)x = \lim_{\xi\to 0}\frac{S_0(\xi)}{\xi}x = \lim_{\xi\to 0}\frac{S(\xi)-S(-\xi)}{2\xi}x =$$

$$= \lim_{\xi\to 0}\frac{S(\xi)-S(0)-\{S(-\xi)-S(0)\}}{2\xi}x$$

also exists, and $G_0(S_0)x=G_0(S)x$. Conversely, if $x\in D\{G_0(S_0)\}$, then by Lemma 1 $x\in D\{S_0'(\alpha)\}$ and $S_0'(\alpha)x=0$. Applying Theorem 2, we obtain that

$$\lim_{\xi\to 0}\frac{S_e(\xi)-S_e(0)}{\xi}x = A\lim_{\xi\to 0}\frac{S_0(\xi+\alpha)-S_0(\alpha)}{\xi}x = 0,$$

and thus exists

$$G_0(S)x = \lim_{\xi\to 0}\frac{S(\xi)-S(0)}{\xi}x = \lim_{\xi\to 0}\frac{S_0(\xi)}{\xi}x = G_0(S_0)x,$$

which ends the proof.

Suppose $S\in V(\alpha, B(X), J, A)$, S_0 is its odd part and C, E are the functions associated with S_0 as in Theorem 1. Introduce the following notations: $G=E'(0)$, $G_1=C'(0)$ and $G_2=C''(0)$, where the domains of the right sides consist of exactly those $x\in X$, for which the respective derivatives exist in the strong topology of X. We define the operators $C'(\xi)$ $(\xi\in R)$ in a similar manner.

Lemma 3. *If* $S\in V(\alpha, B(X), J, A)$, *then for every* $\xi\in R$ *we have* $D(G_0)\subset\subset D\{C'(\xi)\}$, *and* $x\in D(G_1)$ *implies* $G_1x=0$. *Moreover*, $G=iG_0$ *and* $G_2=-G_0^2$.

PROOF. By Lemmas 1 and 2, $x\in D(G_0)$ implies $C'(\xi)x=S_0'(\xi-\alpha)x=-S_0(\xi)G_0x$ for $\xi\in R$. Moreover, $G_1=C'(0)=S_0'(-\alpha)$ and, in view of (3) and (4), $S_0(\eta-\alpha)=$ $=S_0(-\eta-\alpha)$. Hence for $x\in D(G_1)$ we get

$$G_1x=\lim_{\eta\to 0}\frac{S_0(-\alpha+\eta)-S_0(-\alpha)}{\eta}x=$$

$$=\lim_{\eta\to 0}\frac{1/2\,\{S_0(-\alpha+\eta)+S_0(-\alpha-\eta)\}-S_0(-\alpha)}{\eta}x=0.$$

If $x\in D(G_0)$, then $x\in D(G_1)$ and exists

$$Gx=\lim_{\eta\to 0}\left\{\frac{C(\eta)-C(0)}{\eta}x+i\frac{S_0(\eta)}{\eta}x\right\}=iG_0x,$$

while $x\in D(G)$ implies

$$G_0x=\lim_{\eta\to 0}\frac{-i}{2}\left\{\frac{E(\eta)-E(0)}{\eta}x+\frac{E(-\eta)-E(0)}{-\eta}x\right\}=\frac{1}{i}Gx,$$

hence $G=iG_0$.

To prove the last assertion suppose $x\in D(G_0)$, $\eta\neq 0$. Then

$$\frac{C'(\eta)-G_1}{\eta}x=-\frac{S_0(\eta)G_0x}{\eta},\quad \text{thus}\quad D(G_0^2)\subset D(G_2),$$

and $x\in D(G_0^2)$ implies $G_2x=-G_0^2x$. On the other hand, putting $X_1=JX$, $X_0=$ $=(I-J)X$, X is the direct sum of X_1 and X_0. From Theorem 1 it follows that G_0, G and G_2 are identically 0 on X_0, while on X_1 $E(\xi)$ is a strongly continuous group of operators and $C(\xi)$ a strongly continuous cosine operator function (cf. [10]) with $C(0)=J$, the identical operator in X_1. If $\bar{G}_0, \bar{G}, \bar{G}_2$ denote the restrictions of the respective operators to X_1, then there exists a $v>0$ such that $z>v$ implies $z^2\in\varrho(\bar{G}_2)$, $z\in\varrho(\bar{G})\cap\varrho(-\bar{G})$, where ϱ denotes the resolvent set. Hence also $z^2\in\varrho(\bar{G}^2)$, thus

$$(z^2J-\bar{G}^2)D(\bar{G}^2)=X_1=(z^2J-\bar{G}_2)D(\bar{G}_2)$$

the operators on both sides being one-to- one mappings. Therefore we get $D(\bar{G}_2)=$ $=D(\bar{G}^2)$, whence $D(G_0^2)=D(G^2)=D(G_2)$ and $G_2=-G_0^2$.

Remark. In the proof we employed an idea of M. SOVA [10].

Corollary. $D(G_0)$ is dense in X and G_0 is a closed operator. $D(G_0)=X$, i.e. G_0 is bounded if and only if $S(\xi)$ is continuous in the uniform operator topology.

Lemma 4. *If* $S\in V_0(\alpha, B(X), J)$, $y\in X$ *and* $A, B\in R$, *then* $x_{A,B}(y)=\int_A^B S(\eta)y\,d\eta$ *belongs to* $D(G_0)$ *and* $G_0x_{A,B}(y)=\{S(B+\alpha)-S(A+\alpha)\}y$. *Moreover*, $R(J)=R(G_0)$.

PROOF. If E is the group of operators associated with S, then we have

$$\int_A^B S(\eta)\, y\, d\eta = \frac{1}{2i} \int_A^B \{E(\eta)-E(-\eta)\}\, y\, d\eta = \frac{1}{2i}\Big\{\int_A^B E(\eta)\, y\, d\eta + \int_{-A}^{-B} E(\eta) y\, d\eta\Big\}.$$

But then, according to a well-known result of N. DUNFORD, there exists

$$G_0 x_{A,B}(y) = -iGx_{A,B}(y) = -\frac{1}{2}\{E(B)-E(A)+E(-B)-E(-A)\}y =$$

$$= -\{C(B)-C(A)\}y = \{S(B+\alpha)-S(A+\alpha)\}y$$

where C is the function associated with S, occurring in Theorem 1. Hence with $B=-2\alpha$, $A=-\alpha$ for every $y\in X$ we get $S(-\alpha)y=G_0 x_{-\alpha,-2\alpha}(y)$, $R(J)=R\{S(-\alpha)\}\subset R(G_0)$. The converse inclusion was established after Lemma 1.

Corollary. $S\in V_0(\alpha, B(X), J)$ is continuous in the uniform operator topology if and only if $D(G_0^2)=D(G_0)$.

PROOF. If $D(G_0^2)=D(G_0)$, then with the notation of Lemma 3, $X_1=R(G_0)\subset D(G_0)$, thus $D(G_0)=X$ and G_0 is bounded. The converse is evident.

Remark 1. Suppose $S\in V_0(\alpha, B(X), J)$. It can be seen from the proof of Lemma 3 that there is no essential restriction in assuming that $J=S(-\alpha)=I$. To simplify the statements, this assumption will often be made in what follows.

Remark 2. $S\in V(\alpha, B(X), J, A)$ and $S(-\alpha)=I$ hold if and only if $S\in V_0(\alpha, B(X), I)$. Indeed, from Theorem 2 we see that $I=S_0(-\alpha)+S_e(-\alpha)=J+A-AJ$ implies $0=(A-AJ)^2=I-J$, hence $J=I$ and $A=JA=0$. Consequently, we have $S\in V_0(\alpha, B(X), I)$, while the converse is evident.

Theorem 5. *If $S\in V_0(\alpha, B(X), I)$ and γ_n denotes $\frac{\pi}{2\alpha}(4n-1)$ where n is integer, then the spectrum of G_0 is entirely point spectrum, and $\emptyset\neq\sigma(G_0)\subset\{\gamma_n; n=0, \pm 1, \pm 2, \ldots\}$. Moreover, there exist projection operators P_n $(n=0,\pm 1,\pm 2,\ldots)$ in X, for which $P_n X=\{x\in D(G_0);\, G_0 x=\gamma_n x\}\neq\{0\}$ if and only if γ_n is an eigenvalue of G_0. $P_n P_k=\delta_{nk}P_n$, and for $x\in X$ $S(\xi)x=(C,1)-\sum_{n=-\infty}^{\infty}{}' \sin(\gamma_n\xi)P_n x$, $x=(C,1)-\sum_{n=-\infty}^{\infty}{}' P_n x$. For $x\in D(G_0)$ we have $G_0 x=(C,1)-\sum_{n=-\infty}^{\infty}{}' \gamma_n P_n x$ (here $(C,1)-\sum$ denotes Cesàro sum of the first order, and $\sum'$ indicates that summation is extended only for those n, for which $\gamma_n\in\sigma(G_0)$). S is continuous in the uniform operator topology if and only if the number of the eigenvalues of G_0 is finite, and then*

$$S(\xi) = \sum_n{}' \sin(\gamma_n\xi)\cdot P_n,\quad G_0 = \sum_n{}' \gamma_n\cdot P_n,\quad I = \sum_n{}' P_n.$$

PROOF. If E is the group of operators associated with S, then $E(\alpha)=-iI$, and the eigenvalue $-i$ is the unique element of $\sigma\{E(\alpha)\}$. If G is the generator operator of E, then according to the theorems of [6], 16.7. $\emptyset\neq P\sigma(G)\subset\sigma(G)\subset$

$\subset\{\beta_n; n=0, \pm 1, \pm 2, \ldots\}$, where $\beta_n = i\gamma_n$, and thus Lemma 3 implies the statements concerning $\sigma(G_0)$ except $P_\sigma(G_0)=\sigma(G_0)$. However, this will follow from Theorem 6, which implies that all the singularities of the resolvent of G_0 are poles, hence eigenvalues of G_0.

Modifying the ideas in the proof of [6], Theorem 16.7.2., we get that the operators $U(\tau)=e^{i\frac{\pi}{2\alpha}\tau}E(\tau)$ $(\tau\in R)$ in X form a strongly continuous group with period $|\alpha|$. For $x\in X, n=0, \pm 1, \pm 2, \ldots$ we define $P_n x=\frac{1}{\alpha}\int_0^\alpha e^{-in\frac{2\pi}{\alpha}\tau}U(\tau)x\,d\tau$. Then $P_n\in B(X)$, $P_n P_k=\delta_{nk}P_n$, and $P_n X=\{x\in D(G): Gx=\beta_n x\}=\{x\in D(G_0): G_0 x=\gamma_n x\}$. $P_n X\neq\{0\}$ if and only if γ_n is an eigenvalue of G_0, and the strong continuity of $U(\xi)x$ implies for $x\in X$

$$U(\xi)x=(C,1)-\sum_{n=-\infty}^{\infty}{}' e^{in\frac{2\pi}{\alpha}\xi}P_n x,$$

and thus

$$E(\xi)x=(C,1)-\sum_{n=-\infty}^{\infty}{}' e^{i\gamma_n\xi}P_n x, \tag{16}$$

hence $S(\xi)x=(C,1)-\sum_{n=-\infty}^{\infty}{}'\sin(\gamma_n\xi)P_n x$. (16) with $\xi=0$ gives

$$x=(C,1)-\sum_{n=-\infty}^{\infty}{}' P_n x. \tag{17}$$

P_n and $U(\xi)$ commute, hence so do P_n and $S(\xi)$. Therefore $x\in D(G_0)$ implies $P_n x\in D(G_0)$ and $P_n G_0 x=G_0 P_n x=\gamma_n P_n x$. Thus, by (17), we have for $x\in D(G_0)$

$$G_0 x=(C,1)-\sum_{n=-\infty}^{\infty}{}' P_n G_0 x=(C,1)-\sum_{n=-\infty}^{\infty}{}'\gamma_n P_n x.$$

If S is continuous in the uniform operator topology, then the spectral radius of G_0, $r(G_0)\leqq\|G_0\|$, hence the number of the eigenvalues of G_0 is finite. Conversely, if $P\sigma(G_0)$ is finite, then $S(\xi)x$ reduces to a finite sum, and for every $x\in X$ we have $x=\sum' P_n x$, $S(\xi)x=\sum'\sin(\gamma_n\xi)P_n x$, and

$$G_0 x=\lim_{\xi\to 0}\frac{S(\xi)x}{\xi}=\lim_{\xi\to 0}\sum{}'\frac{\sin(\gamma_n\xi)}{\xi}P_n x=\sum{}'\gamma_n P_n x,$$

hence $D(G_0)=X$ and S is continuous in the uniform operator topology. Thus the proof is complete.

A characterization of the generator operator G_0 is given in the following

Theorem 6. *Let G_0 be a closed operator in X with dense domain $D(G_0)$. G_0 is the generator of some $S\in V_0(\alpha, B(X), I)$ $(\alpha\neq 0)$ if and only if there exists a real number $M\geqq 1$ such that 1° and 2° hold:*

1° $(I+zG_0)^{-1}$ *is a bounded operator and* $\|(I+zG_0)^{-n}\| \leqq M$ *for every purely imaginary* z *and for* $n=1, 2, \ldots$

2° *the function* $F(z)=(e^{-i\alpha z}-i)R(z; G_0)$ $\{z\in\varrho(G_0)\}$ *has an analytical continuation* $H(z)$ *on the whole complex plane for which* $\|H(z)\| \leqq Ke^{|\alpha \operatorname{Im} z|}$ *(here* R *denotes the resolvent operator,* $K>0$*).*

In this case the function $S(\xi)$ *generated by* G_0 *is determined uniquely, and* $\|S(\xi)\| \leqq M$ *for* $\xi\in R$.

Remark. A similar theorem has been proved in [9] (Teor. 2.). However, the proof of the sufficiency is not complete there and, in fact, can not dispense with a boundedness condition similar to that in 2°. In addition, our proof seems to be simpler.

PROOF. 1. Suppose $S\in V_0(\alpha, B(X), I)$, G_0 is its generator, E is the group of operators associated with S and $G=iG_0$.

Then 1° follows from Lemma 3 and [6], Theor. 12.3.2. Put $z_k=\frac{i\pi}{2\alpha}(4k-1)$ (k integer), and for $z\neq z_k$, $x\in X$ $U(z)x=(1+ie^{-\alpha z})^{-1}\int\limits_0^\alpha e^{-\xi z}E(\xi)x\,d\xi$. From $E(\alpha)=-iI$ it follows for $x\in D(G)$ that $U(z)Gx=zU(z)x-x$, and by the denseness of $D(G)$ that $R(z; G)=U(z)$.

For $z\in\varrho(G_0)$ we get $R(z; G_0)=iR(iz; G)$, by Lemma 3. Hence $z\neq -iz_k$, $x\in X$ implies $(e^{-i\alpha z}-i)R(z; G_0)x=(1+ie^{-i\alpha z})R(iz; G)x=\int\limits_0^\alpha e^{-i\xi z}E(\xi)x\,d\xi$. Putting $H(z)x=\int\limits_0^\alpha e^{-i\xi z}E(\xi)x\,d\xi$ for every complex z, $H(z)$ is analytic and a continuation of $F(z)$, for $R(z; G_0)$ is holomorphic in $\varrho(G_0)$. Moreover, we have

$$\|H(z)\| \leqq e^{|\alpha \operatorname{Im} z|}\int\limits_0^{|\alpha|}\|E(\xi)\|\,d\xi = Ke^{|\alpha \operatorname{Im} z|},$$

and 2° is also verified.

2. If G_0 satisfies 1°, then $G=iG_0$ generates the strongly continuous group of operators E, for which $\|E(\xi)\| \leqq M$ $(\xi\in R)$. We show that 2° even implies $E(\xi+\alpha)=-iE(\xi)$ for $\xi\in R$.

If $Q(z)=H(-iz)$, then $\|Q(z)\| \leqq Ke^{|\alpha \operatorname{Re} z|}$, and for $\operatorname{Re} z>0$, $x\in X$ we get

$$\frac{Q(z)x}{1+ie^{-\alpha z}} = R(z; G)x = \int\limits_0^\infty e^{-\xi z}E(\xi)x\,d\xi.$$

Suppose $u, v>0$ and $E_2(u)x=\int\limits_0^u\int\limits_0^v E(\xi)x\,d\xi\,dv$, then [6], (6.3.9) gives for $r>0$

$$E_2(u)x = \lim_{B\to\infty}\frac{1}{2\pi i}\int\limits_{r-iB}^{r+iB} e^{zu}\frac{Q(z)x}{1+ie^{-\alpha z}}\cdot z^{-2}\,dz.$$

In the integrand $(1+ie^{-\alpha z})^{-1}$ is bounded, and the absolute convergence of the integral yields with the notation $g(z)=(2\pi i)^{-1}e^{zu}(1+ie^{-\alpha z})^{-1}z^{-2}Q(z)x$ that $E_2(u)x=\int_{r-i\infty}^{r+i\infty} g(z)\,dz$. Making use of 2°, it can be shown after some calculation that $E_2(u)x=2\pi i\left\{\mathrm{Res}\,(g;0)+\sum_{p=-\infty}^{\infty}\mathrm{Res}\,(g;z_p)\right\}$ for $u>|\alpha|$. With the notation

$$Q^*=\left\{\frac{d}{dz}[(1+ie^{-\alpha z})^{-1}Q(z)x]\right\}_{z=0}$$

we get for $u>|\alpha|$

$$E_2(u)x=Q^*+u(1+i)^{-1}Q(0)x+\sum_{p=-\infty}^{\infty}e^{uz_p}(\alpha z_p^2)^{-1}Q(z_p)x.$$

Thus for $u>2\,|\alpha|$ we obtain

$$iE_2(u+\alpha)x-E_2(u)x=(i-1)Q^*+\left(u\cdot\frac{i-1}{i+1}+\frac{i\alpha}{1+i}\right)Q(0)x$$

and, differentiating twice, $E(u+\alpha)x=-iE(u)x$. If $\xi\in R$, then applying $E(\xi-u)$ to both sides we get

$$E(\xi+\alpha)x=-iE(\xi)x\quad(x\in X).$$

If we put $S(\xi)=-\frac{i}{2}\{E(\xi)-E(-\xi)\}$, then $S\in V_0(\alpha,B(X),I)$ by Theorem 1, and Lemma 3 gives that the generator of S is G_0. The uniqueness of S follows from the fact that G determines the group E uniquely. $\|S(\xi)\|\leqq M$ obtains immediately, and the proof is complete.

Since the generator G_0 uniquely determines the function $S\in V_0(\alpha,B(X),I)$, this can be denoted by $S(\xi;G_0)$ if we wish to emphasize the generator.

Theorem 7. *If $S(\xi;G_0)\in V_0(\alpha,B(X),I)$ and $v>0$, then $S(\xi;vG_0)\in V_0\left(\frac{\alpha}{v},B(X),I\right)$, and for every $x\in X$ we have $\lim_{v\to1}S(\xi;vG_0)x=S(\xi;G_0)x$.*

PROOF. The first half of the theorem is clearly true by Theorem 6. If $E(\xi;iA)$ is the group of operators associated with $S(\xi;A)$, then according to the proof of [6], Theorem 12.3.1., for $x\in X$ we obtain $E(\xi;ivG_0)x=\lim_{n\to\infty}\left(I-\frac{\xi}{n}ivG_0\right)^{-n}x=$ $=\lim_{n\to\infty}\left(I-\frac{\xi\cdot v}{n}iG_0\right)^{-n}x=E(\xi\cdot v;iG_0)x$. Since $E(\xi;iG_0)$ is strongly continuous, for $x\in X$ we have $\lim_{v\to1}E(\xi;ivG_0)x=E(\xi;iG_0)x$, and the second half of the theorem follows by Theorem 1.

Remark. Let X be the complex plane, $g_0\in X$, $S(\xi)\zeta=\sin(g_0\xi)\cdot\zeta$ for $\xi\in R$, $\zeta\in X$. If $S(\xi;g_0)\in\bigcup_{\alpha\in R\setminus\{0\}}V_0(\alpha,B(X),I)$, then S is periodic, consequently g_0 is real. Therefore we can not expect that the admissible perturbations of G_0 are much more general than those in Theorem 7.

Theorem 8. *Let H be a Hilbert space and $S\in V_0(\alpha, B(H), I)$ with generator G_0. Then there exists a bounded positive selfadjoint operator Q such that $B=QG_0Q^{-1}$ is a (generally nonbounded) selfadjoint operator with $D(B)=Q\{D(G_0)\}$, and $S(\xi)=Q^{-1}\sin(\xi\cdot B)Q$. Moreover, with the notations of Theorem 5 we have for $x\in H$*

$$S(\xi)x=\sum_{n=-\infty}^{\infty}{}' \sin(\xi\gamma_n)\cdot P_n x,$$

for $x\in D(G_0)$

$G_0x=\sum_{n=-\infty}^{\infty}{}' \gamma_n\cdot P_n x$, *and the projections* $R_n=QP_nQ^{-1}$ *are selfadjoint.*

Proof. If E is the group associated with S, then $\sup_{\xi\in R}\|E(\xi)\|\leqq K$ implies, according to [8], that there exists a bounded selfadjoint operator Q with $\frac{1}{K}\cdot I\leqq Q\leqq K\cdot I$ such that $QE(\xi)Q^{-1}$ is unitary for $\xi\in R$. Then there exists a selfadjoint B such that $E(\xi)=Q^{-1}e^{i\xi B}Q$, and thus $S(\xi)=Q^{-1}\sin(\xi B)Q$, where clearly $\sin(\xi B)\in V_0(\alpha, B(H), I)$ and B is its generator. If $Qx\in D(B)$, then there exists $G_0x=\lim_{\xi\to 0}\frac{1}{\xi}Q^{-1}\sin(\xi B)Qx=Q^{-1}BQx$, thus $Q^{-1}\{D(B)\}\subset D(G_0)$. If $Q^{-1}x\in D(G_0)$, then $Bx=\lim_{\xi\to 0}\frac{1}{\xi}\sin(\xi B)x=\lim_{\xi\to 0}\frac{1}{\xi}QS(\xi)Q^{-1}x=QG_0Q^{-1}x$, thus $Q\{D(G_0)\}\subset D(B)$, hence $D(B)=Q\{D(G_0)\}$ and $B=QG_0Q^{-1}$ as stated.

B and G_0 are similar, hence $\sigma(B)=\sigma(G_0)$ and $P\sigma(B)=P\sigma(G_0)$ (cf. [4], Problem 60.). The spectral representation of B gives for $x\in D(B)$ that $Bx=\sum_{n=-\infty}^{\infty}{}' \gamma_n\cdot R_n x$, where R_n is a selfadjoint projection ($n=0, \pm 1, \pm 2, \dots$) with $R_nR_k=\delta_{nk}R_n$. From this we obtain for $x\in D(G_0)$ that $G_0x=\sum_{n=-\infty}^{\infty}{}' \gamma_n\cdot Q^{-1}R_nQx$, and for $x\in H$ that $S(\xi)x=\sum_{n=-\infty}^{\infty}{}' \sin(\xi\gamma_n)Q^{-1}R_nQx$. Denote $Q^{-1}R_nQ$ by Q_n, we shall show that $Q_n=P_n$.

If $x\in Q_nH$, then $S(\xi)x=\sin(\xi\cdot\gamma_n)x$ and $G_0x=\gamma_nx$, thus, by Theorem 5, $x\in P_nH$. Conversely, $x\in P_nH$ implies $iG_0x=i\gamma_n\cdot x$, hence $E(\xi)x=e^{i\gamma_n\xi}x$ and

$$\sin(\gamma_n\xi)\cdot x=S(\xi)x=\sum_{k=-\infty}^{\infty}{}' \sin(\xi\cdot\gamma_k)Q_kx.$$

Applying P_n to both sides, we get because of $Q_kH\subset P_kH$ that $\sin(\gamma_n\xi)x=\sin(\gamma_n\xi)Q_nx$, and hence $x\in Q_nH$. Thus $P_n=Q^{-1}R_nQ$ for every integer n and the proof is complete.

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TECHNICAL UNIVERSITY, BUDAPEST
ADDRESS: B. NAGY, VÉCSEY U. 28/A. I. 5.
1204 BUDAPEST, HUNGARY

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