The distribution of additive functions on the set of divisors

By I. KÁTAI (Budapest)

1. Let $\tau(n)$ denote the number of divisors of n. If n has the prime-factorization

(1.1) $$n = p_1^{\alpha_1} \dots p_r^{\alpha_r},$$

then

$$\tau(n) = (\alpha_1+1) \dots (\alpha_r+1).$$

Let $f(m)$ be an arbitrary completely additive function, i.e. the relation

$$f(kl) = f(k)+f(l)$$

is satisfied for every pair k, l of integers.

First we give a proof for the relation

(1.2) $$\frac{1}{\tau(n)} \sum_{d|n} \left(f(d)-\frac{1}{2}f(n)\right)^2 = \sum_{i=1}^{r} c_i f^2(p_i),$$

where

(1.3) $$c_i = \frac{\alpha_i(\alpha_i+2)}{12}.$$

We give a probabilistical proof of (1.2).

Let (Ω, A, P) be a probability space and $\xi_1, \xi_2, \dots, \xi_r$ be a sequence of independent random variables with the distribution

(1.4) $$P\left(\xi_j = \left(\beta-\frac{\alpha_j}{2}\right)f(p_j)\right) = \frac{1}{\alpha_j+1} \quad (\beta = 0, 1, \dots, \alpha_j;\ j = 1, \dots, r).$$

Let

$$\eta = \xi_1+\dots+\xi_r.$$

We observe that η takes the values $f(d)-\frac{1}{2}\cdot f(n)$ with equal probability for every divisor d of n:

(1.5) $$P\left(\eta = f(d)-\frac{1}{2}\cdot f(n)\right) = \frac{1}{\tau(n)}.$$

We calculate the dispersion of η in two ways, and obtain the formula desired. Since the mean values

$$M\xi_j = \frac{1}{\alpha_j+1} \sum_{\beta=0}^{j} \left(\beta - \frac{\alpha_j}{2}\right) f(p_j) = 0,$$

and for the second moments

$$M\xi_j^2 = \frac{1}{\alpha_j+1} \sum_{\beta=0}^{\alpha_j} \left(\beta - \frac{\alpha_j}{2}\right)^2 \cdot f^2(p_j) = c_j f^2(p_j),$$

therefore

$$M\eta^2 = M\xi_j^2 = \sum_{j=1}^{r} c_j f^2(p_j).$$

Furthermore, from (1.5) we get

$$M\eta^2 = \frac{1}{\tau(n)} \sum_{d|n} \left(f(d) - \frac{1}{2} f(n)\right)^2,$$

and this finishes the proof.

2. Let $\tau(n, \alpha, \beta)$ denote the number of those divisors d of n, for which $n^\alpha \leqq d < n^\beta$; $0 \leqq \alpha < \beta \leqq 1$.

Taking $f(d) = \log d$, from (1.2) we get

$$\frac{1}{\tau(n)} \sum_{d|n} \left(\frac{\log d}{\log n} - \frac{1}{2}\right)^2 = \sum_{i=1}^{r} c_i \varepsilon_i^2, \tag{2.1}$$

where

$$c_i = \frac{\alpha_i(\alpha_i+2)}{12}, \quad \varepsilon_i = \frac{\log p_i}{\log n}.$$

Let $P(n)$ denote the greatest prime power divisor of n. From (2.1) we immediately get the following assertion, which was proved in [1].

Theorem 1. *For the infinite sequence $n_1 < n_2 < \dots$ of integers the relation*

$$\tau\left(n_k, \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) / \tau(n_k) \to 1 \quad (k \to \infty) \tag{2.2}$$

for every positive ε satisfies if and only if

$$\log P(n_k) / \log n_k \to 0 \quad (k \to \infty). \tag{2.3}$$

Let now $f(n)$ be positive for every n, and $N(n; \alpha, \beta)$ the number of those divisors d, for which

$$\alpha \leqq f(d)/f(n) < \beta.$$

Let

$$\mu_n = \frac{\max_{i=1,\dots,r} f(p_i^{\alpha_i})}{f(n)}.$$

Theorem 2. *Let* $n_1 < n_2 < \dots$ *be an infinite sequence of integers. The relation*

$$N\left(n_k, \frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right)/\tau(n_k) \to 1 \quad (k \to \infty)$$

for every positive ε, *if and only if* $\mu_{n_k} \to 0$ $(k \to \infty)$.

The proof is very simple, and so we omit it.

We now consider a simple consequence of this theorem. Let us suppose that

$$\sum_p \frac{f(p)}{p} = \infty, \quad 0 \leqq f(p) = O(1). \tag{2.4}$$

The Turán—Kubilius inequality gives that

$$\sum_{n \leqq x} \left(f(n) - \sum_{p \leqq x} \frac{f(p)}{p} \right)^2 \ll x \sum_{p \leqq x} \frac{f^2(p)}{p}, \quad f(n) \to \infty \quad (n \to \infty)$$

except for a set of integers having zero density. Then $\mu_n \to 0$ on the same set, and

$$N\left(n, \frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right)/\tau(n) \to 1 \quad (n \to \infty)$$

except a set of zero density.

3. Now we shall prove that on some assumptions the theorem of central-limit distribution is valid too.

We need a Lemma due to Ljapunov.

Lemma. *Let* $X_1, \dots, X_r$ *be independent random variables, let*

$$MX_i = 0, \quad MX_i^2 = \sigma_i^2, \quad MX_i^3 = \beta_i \quad (i = 1, \dots, r),$$

$$S_r^2 = \sum_{i=1}^{r} \sigma_i^2, \quad B_2 = \frac{1}{r} S_r^2, \quad B_3 = \frac{1}{r} \sum_{i=1}^{r} \beta_i,$$

and

$$\bar{F}(x) = P(X_1 + \dots + X_r < x \cdot S_r).$$

Then

$$|\bar{F}(x) - \Phi(x)| < \frac{c}{\sqrt{r}} \frac{B_3}{B_2^{3/2}},$$

where c *is an absolute constant and*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du.$$

Now we use this lemma by choosing $X_j = \xi_j$. Then

$$S_r^2 = \sum_{i=1}^{r} \frac{\alpha_i(\alpha_i + 2)}{12} \cdot f^2(p_i).$$

Furthermore

$$M|\xi_i|^3 = \left(\frac{1}{(\alpha_i+1)} \sum_{\nu=0}^{\alpha_i} \left|\nu - \frac{\alpha_i}{2}\right|^3\right) |f^3(p_i)| = \tau_i |f(p_i)|^3.$$

As we can see easily that $c_1\alpha_i^3 \leqq \tau_i \leqq c_2\alpha_i^3$, c_1, c_2 are positive constants.

Therefore

$$\frac{B_3}{B_2}\frac{3}{2} \leqq c\sqrt{r} \cdot \frac{\sum_{i=1}^{r} |f(p_i^{\alpha_i})|^3}{\left(\sum_{i=1}^{r} |f(p_i^{\alpha_i})|^2\right)^{3/2}}.$$

We need to estimate the right hand side of the inequality. Let

$$\text{(3.1)} \qquad \begin{cases} \Lambda = \sum_{i=1}^{r} |f(p_i^{\alpha_i})|, \\ \Delta = \left(\max_{i=1,\dots,r} |f(p_i^{\alpha_i})|\right)/\Lambda, \\ \theta_i = |f(p_i^{\alpha_i})|/\Lambda \quad (i=1,\dots,r). \end{cases}$$

Then

$$(K \stackrel{\text{def}}{=}) \frac{\sum_{i=1}^{r} |f(p_i^{\alpha_i})|^3}{\left(\sum_{i=1}^{r} |f(p_i^{\alpha_i})|^2\right)^{3/2}} = \frac{\sum_{i=1}^{r} \theta_i^3}{\left(\sum_{i=1}^{r} \theta_i^2\right)^{3/2}}.$$

Using elementary analysis we can see that

$$K \leqq c\Delta^{1/2}$$

on the conditions

$$0 \leqq \theta_i \leqq \Delta, \quad \sum \theta_i = 1.$$

Hence we get the following

Theorem 3. *Let* $N_n(x)$ *denote the number of those divisors* d *of* n *for which*

$$\text{(3.2)} \qquad f(d) < \frac{f(n)}{2} + x\left(\sum_{i=1}^{r} \frac{\alpha_i(\alpha_i+2)}{12} f^2(p_i)\right)^{1/2}.$$

We assume that (2.4) *holds. Then*

$$\text{(3.3)} \qquad \left|N_n(x) - \frac{\tau(n)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du\right| < c\Delta^{1/2} \cdot \tau(n),$$

where c *is an absolute constant and* Δ *is defined in* (3.1).

This relation was proved for $f(d) = \log d$, and some special integers n by *Babaev* [2].

Let us suppose that

$$f(p) = O(1),$$

and that the sum

$$A(x) = \sum_{p<x} \frac{|f(p^\alpha)|}{p^\alpha}$$

tends to infinity for $x \to \infty$.

Let furthermore

$$g(n) = \sum_{p^\alpha \| n} |f(p^\alpha)|.$$

By the Turán—Kubilius inequality we have

$$\sum_{n \leqq x} (g(n) - A(x))^2 \ll x \sum_{p \leqq x} \frac{f^2(p^\alpha)}{p^\alpha} \ll x \cdot A(x).$$

Hence it follows that

$$g(n) > \frac{1}{2} A(x)$$

for all but $o(x)$ of the integers $n \leqq x$. Furthermore

$$\max_{p^\alpha \| n} |f(p^\alpha)| \leqq c\alpha.$$

Let $\omega(x)$ be an arbitrary function tending to infinity monotonically as $x \to \infty$. As we can see easily that

$$\max_{p^\alpha \| n} |f(p^\alpha)| \leqq \omega(x)$$

for all but $\sigma(x)$ of the integers $n \leqq x$. Therefore

$$\Delta \leqq 2 \frac{\omega(x)}{A(x)}$$

for almost all $n (\leqq x)$.

Hence we get almost immediately the following assertion.

Theorem 4. *Let $f(p) = 0(1)$ and assume that*

$$A(x) = \sum_{p<x} \frac{|f(p^\alpha)|}{p^\alpha}$$

tends to infinity as $x \to \infty$. Then for all integer n except at most a set of zero density the inequality

$$\left| \frac{N_n(y)}{\tau(n)} - \Phi(y) \right| \leqq c \frac{\omega(n)}{A(n)}$$

holds uniformly in y; $\omega(n)$ is an arbitrary function which tends to infinity monotonically.

We can prove similar assertions by using the method of characteristic functions.

4. We can prove similar theorems for the distribution of additive functions on the set of solutions of the equation $n = x_1 \cdot x_2 \ldots \cdot x_k$.

We state only a special result without proof (see [1]). Let $\tau_k(n)$ denote the number of positive integer solutions of the equation $n=x_1...x_k$, and $\tau_k(n, \alpha, \beta)$ the number of those solutions for which

$$n^{\alpha_i} \leqq x_i < n^{\beta_i} \quad (i = 1, ..., k-1)$$

$$\underline{\alpha} = (\alpha_1, ..., \alpha_{k-1}), \quad \underline{\beta} = (\beta_1, ..., \beta_{k-1}).$$

Theorem 5. *Let* $(1 \leqq)\ n_1 < n_2 < ...$ *be an infinite sequence of integers, and let*

$$\underline{\gamma} = \left(\frac{1}{k} - \varepsilon, ..., \frac{1}{k} - \varepsilon\right); \quad \underline{\delta} = \left(\frac{1}{k} + \varepsilon, ..., \frac{1}{k} + \varepsilon\right).$$

The relation

$$\tau_k(n_j, \underline{\gamma}, \underline{\delta})/\tau_k(n_j) \to 1 \quad (j \to \infty)$$

holds for every positive ε *if and only if*

$$\frac{\log P(n_j)}{\log n_j} \to 0 \quad (j \to \infty).$$

References

[1] I. Kátai, On the distribution of the solutions of special Diophantine equations, *Matematikai Lapok*, **20** (1969), 117—122 *(In Hungarian)*.

[2] G. Babaev, Distribution of integer points on algebraic surfaces, *Duschanbe*, 1966.

[3] C. D. Esseen, Fourier analysis of distribution functions, *Acta Math.*, **77** (1945), 1—125.

(Received January 1, 1975.)