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On a conjecture concerning additive number theoretical functions

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Abstract. The main result of the paper is as follows: If f is completely additive and $f(2n+4k+1)-f(n)$ is monotonic from some number on, then $f(n)=c\log n$.

In 1946 Erdős [2] proved the following theorem:

Theorem 1 (Erdős). *If the real valued additive function f is monotonic, then $f(n)=c\log n$.*

As a possible generalization of this result I proposed the following conjecture.

Conjecture. *Let f be an additive function. If $f(an+b)-f(cn+d)$ is monotonic from some number on, then $f(n)=c\log n$ for all n coprime to $ac(ad-bc)$.*

If f is bounded, then $f(an+b)-f(cn+d)$ is convergent and the conjecture is true by a theorem of Elliott [1]. In [3] we proved some special cases of the conjecture, including the following theorem.

Theorem 2. *Let f be an additive function and let a and b be different integers. If $f(n+a)-f(n+b)$ is monotonic, or it is of constant sign from some number on, then $f(n)=c\log n$ for all n coprime to $a-b$. If f is completely additive, then $f(n)=c\log n$ for all n.*

Here we prove the conjecture in certain further special cases.

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Theorem. *Let f be a completely additive function.*
(i) *If* $A-2B \not\equiv -1 \mod 4$ *and*

$$f(2n+A)-f(n+B) \tag{1}$$

is monotonic from some number on, then $f(n)=c\log n$ *for all* n.
(ii) If

$$f(2n+1)-f(n-1) \tag{2}$$

is monotonic, then $f(n)=c\log n$ for all n.

Proof of the Theorem.
(i) Let us replace n by $n-B$ in (1). So

$$f(2n+A-2B)-f(n) \tag{3}$$

is monotonic from some number on. We may assume that it is increasing.
If $2|A-2B$, then by Theorem 2 $f(n)=c\log n$ for all n.
Otherwise $A-2B=4k+1$ with some k. So

$$f(2n+4k+1)-f(n) \tag{4}$$

is increasing from some number on. By comparing the value of (4) at the numbers $n-(k+1)$ and $n^2-(k+1)^2$ we obtain

$$f(2n^2-2k^2-1)-f(n^2-(k+1)^2 \geq f(2n+2k-1)-f(n-(k+1)).$$

By comparing its values at $n-3k$ and $2n^2-2k^2-1$ we get

$$f\left(4n^2-(2k-1)^2\right)-f(2n^2-2k^2-1) \geq f(2n-2k+1)-f(n-3k).$$

By adding these inequalities we obtain

$$f(n-3k)-f(n+k+1) \geq 0$$

from some number on. By Th.2. $f(n)=c\log n$ for all n.

Background of the proof. A has to be odd and we may assume that B is even (otherwise replace n by $n-1$ in (1)). Therefore $A-2B\equiv 1 \mod 4$ yields $A\equiv 1 \mod 4$.
Let us replace n by $2n^2+d$ in (1). We have

$$f(4n^2+2d+A)-f(2n^2+d+B) \geq f(2N_1+A)-f(N_1+B). \tag{5}$$

By the suitable choice of d we have $2d+A=-u^2$ with some odd integer u. So

$$f(4n^2+2d+A)=f(2n+u)+f(2n-u)$$

on the left hand side of (5). Let us choose N_1 such that $2N_1 + A = 2n + u$ appears also on the right hand side. So (5) transforms into

$$(6) \quad f(2n+u) + f(2n-u) - f(2n^2+d+B) \geq f(2n+u) - f\left(n + \frac{u-A}{2} + B\right).$$

To gain $f(2n^2+d+B)$ on the left and $f(2n-u)$ on the right of an inequality let us compair (1) replacing n by N_2 ans N_3 such that $2N_2+A = 2n^2+d+B$ and $2N_3 + A = 2n - u$. So we have

$$(7) \quad f(2n^2+d+B) - f\left(n^2 + \frac{d+B-A}{2} + B\right) \geq f(2n-u) - f\left(n - \frac{u+A}{2} + B\right).$$

Here d has to be odd to get integers in the arguments.

Adding (6) and (7) we have

$$f\left(n - \frac{u+A}{2} + B\right) + f\left(n + \frac{u-A}{2} + B\right) \geq f\left(n^2 + \frac{d+B-A}{2} + B\right).$$

If $\frac{d+B-A}{2} + B = -v^2$ with some integer v and $v = \frac{u-A}{2} + B$, then

$$f\left(n - \frac{u+A}{2} + B\right) - f(n-v) \geq 0,$$

i.e. by Theorem 2 $f(n) = c\log n$ for all n.

We are looking for an odd integer u. As $d = \frac{-u^2-A}{2}$ and $v^2 = \frac{A-3B-d}{2}$, so u must be the solution of

$$A - 2B = u - 2v = u \pm \sqrt{u^2 + 3A - 6B}.$$

For odd A and even B, $u = \frac{A-2B-3}{2}$ is a satisfactory choice for u.

(ii) We compare the value of the function (2) at n and $2n^2 + 2n$, then at n and $n^2 + n - 1$. By adding the resulting inequalities

$$f\left[(2n+1)^2\right] - f(2n^2+2n-1) \geq f(2n+1) - f(n-1)$$

and

$$f(2n^2+2n-1) - f(n^2+n-2) \geq f(2n+1) - f(n-1),$$

we obtain $f(n-1) - f(n+2) \geq 0$, hence by Theorem 2 we infer that $f(n) = c\log n$ for all n.

References

[1] P. D. T. A. Elliott, Arithmetic functions and integer products, Grundlehren der mathematischen Wissenschaften, Bd. 272, *Springer Verlag*, *New York–Berlin*, 1985, MR 86j: 11095.

[2] P. Erdős, On the distribution function of additive functions, *Ann. of Math. (2)* **47** (1946), 1–20, MR 7–416.

[3] K. Kovács, On a generalization of an old theorem of Erdős, *Studia Sci. Math. Hung.* **29** (1994), 209–212.

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