Topologically arithmetical rings of continuous functions

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In memoriam of Andor Kertész

The closed ideals $\mathfrak{a}, \mathfrak{b}, \ldots$ in a topological ring R form a lattice ordered semigroup $V_\Gamma(R)$ with respect to their intersection $\cap$, to the closure $\mathfrak{a}\cup\mathfrak{b}=\Gamma(\mathfrak{a}+\mathfrak{b})$ of their sum $\mathfrak{a}+\mathfrak{b}$ and to the closure $\mathfrak{a}\circ\mathfrak{b}=\Gamma(\mathfrak{a}\cdot\mathfrak{b})$ of their products. As in algebraic number theory, we call the structure of $V_\Gamma(R)$ the arithmetic of $\boldsymbol{R}$.

R is called *topologically arithmetical* if the lattice $V_\Gamma(R)$ is distributive.

An example is provided by the Banach algebra $A=\mathscr{C}_{\boldsymbol{C}}(X)$ of all continuous functions on a compact Hausdorff space X with values in the field $\boldsymbol{C}$ of complex numbers under the supremum norm:

$$\|f\|_\infty = \sup\{|f(x)|;\ x\in X\} \quad \text{for} \quad f\in A. \tag{1}$$

It is well known (e.g. NAIMARK [1], Theorem 16.3) that in this case the closed ideals $\mathfrak{a}$ correspond one-to-one to the closed subsets A of X by

$$A = \{x\in X;\ f(x) = 0 \text{ for all } f\in\mathfrak{a}\}. \tag{2}$$

Here $\mathfrak{a}\cap\mathfrak{b}$ and $\mathfrak{a}\cup\mathfrak{b}$ correspond to $A\cup B$ and $A\cap B$ respectively and the product $\mathfrak{a}\circ\mathfrak{b}$ equals $\mathfrak{a}\cap\mathfrak{b}$. The ring $\boldsymbol{P}=\boldsymbol{C}[[\omega]]$ of formal power series in one indeterminate ω is not normable (ALLAN [1]) in such a way that it becomes a $\boldsymbol{C}$-Banach algebra, that means so that $\boldsymbol{P}$ is complete with respect to the norm; but $\boldsymbol{P}$ is a complete, locally convex algebra (MICHAEL [1]) under the sequence of seminorms:

$$p_n\alpha = |a_0|+\ldots+|a_n| \quad \text{for} \quad \alpha = \sum_{i\geqq 0} a_i\omega^i, \quad a_i\in\boldsymbol{C}. \tag{3}$$

They provide a neighborhood basis for the zero series 0 by the sets

$$\mathfrak{A}_n(0) = \{\alpha\in\boldsymbol{C}[[\omega]];\ p_n\alpha < \varepsilon\}, \quad n = 0, 1, 2, \ldots, \ \varepsilon > 0. \tag{4}$$

The ideals of $\boldsymbol{P}$ are the powers of its radical $\omega\boldsymbol{P}$, and they are closed.

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The combination of the wo examples above leads to the $\boldsymbol{P}$-algebra $\mathscr{C}_{\boldsymbol{P}}(X)$ of the continuous $\boldsymbol{P}$-valued functions on a compact Hausdorff space, a complete locally convex algebra R under the sequence of the seminorms

$$q_n(f) = \sup\{p_n(f(x));\ x\in X\} \quad \text{for} \quad f\in R, \quad 0 \leqq n\in \boldsymbol{Z}. \tag{5}$$

We will prove that $R=\mathscr{C}_{\boldsymbol{P}}(X)$ is topologically arithmetical and we will develop its arithmetic: Any closed ideal in R is the intersection

$$\mathfrak{a} = \bigcap\{\mathfrak{m}^{\mu(\mathfrak{a},\mathfrak{m})};\ \mathfrak{m}\in\mathfrak{X}\} \tag{6}$$

where $\mathfrak{X}$ is the set of maximal ideals in R, a compact Hausdorff space in its hull-kernel topology and homeomorphic to X, (Naimark [1], III. § 15 and § 16). The exponents, if taken maximal, are uniquely determined by $\mathfrak{a}$ and $\mu(\mathfrak{a}, -)$ is an upper semicontinuous function (e.g. Bourbaki [1], IV. 6.2) on $\mathfrak{X}$ with values in the set $\bar{N}^\circ$ if non-negative rational integers together with ∞. Here $\mathfrak{m}^\circ$ means R and $\mathfrak{m}^\infty$ means the intersection $\mathfrak{n}_{\mathfrak{m}}$ of all powers $\mathfrak{m}^k$ of the maximal ideal $\mathfrak{m}$. The lattice ordered semigroup $V_\Gamma(R)$ of the closed ideals in $R=\mathscr{C}_{\boldsymbol{P}}(X)$ is isomorphic to the lattice ordered semigroup $\mathfrak{S}$ of upper semicontinuous $\bar{N}^\circ$-valued functions $\alpha, \beta, \ldots$ under

$$\begin{cases}(\alpha+\beta)(\mathfrak{m}) = \alpha(\mathfrak{m})+\beta(\mathfrak{m}) \quad \text{and} \quad (\alpha\cap\beta)(\mathfrak{m}) = \sup\{\alpha(\mathfrak{m}),\ \beta(\mathfrak{m})\}\\ \text{and} \quad (\alpha\cup\beta)(\mathfrak{m}) = \inf\{\alpha(\mathfrak{m}),\ \beta(\mathfrak{m})\} \quad \text{for} \quad \mathfrak{m}\in\mathfrak{X}.\end{cases} \tag{7}$$

(Theorem 3.1).

The sum of finitely many closed ideals in R is closed (Theorem 2.2) and the closure $\mathfrak{a}\circ\mathfrak{b}=\Gamma(\mathfrak{a}\cdot\mathfrak{b})$ of the product $\mathfrak{a}\cdot\mathfrak{b}$ of the closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ satisfies

$$\mathfrak{a}\cdot\mathfrak{b}\subseteqq\mathfrak{a}\circ\mathfrak{b}\subseteqq\bigcap\{\mathfrak{a}\cdot\mathfrak{b}+\omega^n R\};\ n\in N\}, \quad N = 1, 2, 3, \ldots \tag{8}$$

(Theorem 2.4). The ring R is a topologically principal ideal ring, i.e. $\mathfrak{a}=\Gamma(Rf)$ for $\mathfrak{a}\in V_\Gamma(R)$, if and only if the compact Hausdorff space X is perfectly normal (Theorem 4.1).

The Gelfand—Naimark Theorem (Naimark [1], Theorem III. 16.2.1) characterizes, up to isometric isomorphisms, the $\boldsymbol{C}$-algebras $\mathscr{C}_{\boldsymbol{C}}(X)$, X a compact Hausdorff space, as the commutative B^*-algebras A with an identity element e. A characterization of the $R=\mathscr{C}_{\boldsymbol{P}}(X)$ above as commutative, topologically arithmetical, complete, locally convex algebras with e will be given in a forthcoming paper, using the classical result that in the B^*-algebra $\mathscr{C}_{\boldsymbol{C}}(X)$ every closed ideal is the kernel of its hull (Naimark [1], Theorem III. 16, 3) and observing that by Corollary 2.9 the factor R-module $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an one-dimensional $\boldsymbol{C}$-linear space because otherwise the intervale $[\mathfrak{m}^{n+1}, \mathfrak{m}^n]$ in $V_\Gamma(R)$ would contain a projective root, in contradiction to the distributivity of $V_\Gamma(R)$.

1.

The ring $\boldsymbol{P}$ of formal power series $\alpha=\Sigma a_i\omega^i$ in one indeterminate ω with non-negative exponents i and coefficients a_i in the field $\boldsymbol{C}$ of complex numbers becomes a complete, locally convex $\boldsymbol{C}$-algebra under the sequence

$$p_n : \alpha \to |a_0|+|a_1|+\ldots+|a_n| \quad \text{for} \quad \alpha\in\boldsymbol{P} \tag{1}$$

of seminorms p_n, $n=0, 1, 2, \ldots$, because $p_n(\alpha \cdot \beta) \leqq (p_n\alpha)(p_n\beta)$; a neighborhood basis for the zero series is given by the sets

$$(2) \qquad U_{n,\varepsilon} = \{\alpha \in P;\ p_n\alpha < \varepsilon\}, \quad \text{where} \quad \varepsilon > 0 \quad \text{and} \quad n = 0, 1, 2, \ldots.$$

The zero sets of the seminorms are the ideals

$$(3) \qquad \{\alpha \in \boldsymbol{P};\ p_n\alpha = 0\} = \{\alpha \in \boldsymbol{P};\ \text{ord}\, \alpha > n\},$$

where the order of a power series is defined by

$$(4) \qquad \text{ord}\, \alpha = \min\{i;\ a_i \neq 0\} \quad \text{in} \quad \alpha = \Sigma a_i \omega^i.$$

The radical of $\boldsymbol{P}$ is the ideal $\omega\boldsymbol{P}$ and its powers are the only ideals in $\boldsymbol{P}$.

If X is a compact Hausdorff space then the set of all continuous $\boldsymbol{P}$-valued functions on X becomes a complete, locally convex $\boldsymbol{P}$-algebra $R = \mathscr{C}_P(X)$ under the following sequence of supremum norms

$$(5) \qquad q_n f = \sup\{p_n(f(x));\ x \in X\} \quad \text{for} \quad f \in R, \quad n = 0, 1, 2, \ldots.$$

Here $q_n f$ is a real number because $p_n f$ is a real-valued continuous function on the compact Hausdorff space X. The inequality $q_n(f \cdot g) \leqq (q_n f) \cdot (q_n g)$ is simple to prove. The completeness of R follows from the fact that the Cauchy sequences in R are uniformly convergent sequences of continuous functions on X.

To get a correlation in between the closed ideals in R and the descending sequences of closed subsets in X set

$$(6) \qquad \mu(\mathfrak{a}, x) = \min\{\text{ord}(f(x));\ f \in \mathfrak{a}\}$$

where $\mathfrak{a}$ is a closed ideal in R and x a point in X. It is convenient to denote the closure of a subset $\mathfrak{M}$ in R or X by $\Gamma\mathfrak{M}$. Then

$$(7) \qquad \mathfrak{a} \circ \mathfrak{b} = \Gamma(\mathfrak{a} \cdot \mathfrak{b})$$

is the smallest closed ideal in R containing the product $\mathfrak{a} \cdot \mathfrak{b}$ of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R. The function μ satisfies

$$(8) \qquad \mu(\mathfrak{a} \circ \mathfrak{b}, x) = \mu(\mathfrak{a}, x) + \mu(\mathfrak{b}, x)$$

for closed ideals $\mathfrak{a}$ and $\mathfrak{b}$. This equation is a consequence of the equation $\text{ord}(\alpha \cdot \beta) = \text{ord}\, \alpha + \text{ord}\, \beta$ in $\boldsymbol{P}$ and of the implication $f = \lim f_n$, $\text{ord}(f_n(x)) \geqq m$ for $n = 1, 2, 3, \ldots \Rightarrow \text{ord}(f(x)) \geqq m$, together with the continuity of the mapping

$$(9) \qquad f \to f(x) \quad \text{for} \quad f \in R$$

from R to $\boldsymbol{P}$ for fixed $x \in X$.

Now the continuity of the functions in R implies that the subsets

$$(10) \qquad A_i = \{x \in X;\ f(x) \in \omega^{i+1}\boldsymbol{P} \text{ for all } f \in \mathfrak{a}\}, \quad i = -1, 0, 1, 2, \ldots,$$

are closed, even if $\mathfrak{a}$ is a not necessarily closed ideal in R. Because $\mathfrak{a}$ as an P-module contains with f the function ωf also, the A_i form a descending sequence of closed subsets of X: Let $f \in \mathfrak{a}$ and $x \in A_{i+1}$; then $\omega f(x) \in \omega^{i+2}\boldsymbol{P}$ implies $f(x) \in \omega^{i+1}\boldsymbol{P}$.

As it will be proved very soon, a closed ideal $\mathfrak{a}$ is determined by this sequence. Therefore we introduce the mapping

$$\varphi : \mathfrak{a} \to \{A_i;\ i = -1, 0, 1, \ldots\} \tag{11}$$

of the ideals $\mathfrak{a}$ in R into the set of descending sequences of closed subsets A_i of X beginning with $A_{-1}=X$. On the other hand, if we start with a sequence $\{A_i; i\}$ as above we define by

$$\chi : \{A_i;\ i = -1, 0, 1, 2, \ldots\} \to \{f \in R;\ x \in A_i \Rightarrow f(x) \in \omega^{i+1}\boldsymbol{P};\ i\} \tag{12}$$

a mapping of the sequences $\{A_i; A_i = \Gamma A_i \supseteqq A_{i+1},\ i=-1, 0, 1, 2, \ldots\}$ of subsets A_i of X to subsets of R. These subsets of R are ideals in R which are closed in virtue of the continuity of the mapping (9).

The key for the ideal theory in R is the statement that $\chi\varphi$ and $\varphi\chi$ are the identical mappings of the set of closed ideals in R and of the set of descending chains of closed subsets of X respectively: For a given descending chain $\{A_i; i\}$ as in (12) set $\{A_i'; i\} = \varphi\chi\{A_i; i\}$. Then the inclusion $A_i \subseteqq A_i'$ for all i is clear by a comparison of (10) with (12). If $x_0 \notin A_{i_0}$ then by a theorem of Urysohn (e.g. BOURBAKI [1], Proposition IX. 4.1.1) there exists a function $\alpha \in \mathscr{C}_{\boldsymbol{R}}(X)$ which is equal to 0 at every point in $A_{i_0} = \Gamma A_{i_0}$ and satisfies $\alpha(x_0)=1$. Then the function $\alpha \cdot \omega \in \mathscr{C}_{\boldsymbol{P}}(X)$ is contained in χ $(\{A_i; i\})$ but $\alpha(x_0)\omega^i \notin \omega^{i+1}\boldsymbol{P}$. That shows $x_0 \notin A_{i0}'$ and finishes the proof of

$$\varphi\chi(\{A_i;\ i = -1, 0, 1, \ldots\} = \{A_i;\ i = -1, 0, 1, \ldots\}. \tag{13}$$

Let $\mathfrak{a}$ be a (not necessarily closed) ideal in R and $\{A_i; i\} = \varphi\mathfrak{a}$. Let $f \in \chi\varphi\mathfrak{a}$, i.e. for this f the following implication is valid

$$x \in A_i \Rightarrow f(x) \in \omega^{i+1}\boldsymbol{P} \quad \text{for all} \quad i. \tag{14}$$

Because $A_{-1}=X$ there exists $a_{-1} \in \mathfrak{a}$ with $f \equiv a_{-1}$ (modulo $\omega^{-1+1}R$). We want to show that for every $n \in \mathbf{N}^{\circ}$ there exists an $a_n \in \Gamma\mathfrak{a}$ with $q_n(f-a_n)=0$, because that would imply $f \in \Gamma\mathfrak{a}$. Take this statement for n as an induction assumption and set $f_n = f - a_n$. Select a point $x_0 \in X$. If $f_n(x_0) \in \omega^{n+2}\boldsymbol{P}$ set $g_0 = \text{const} = 0$ on X. If $f_n(x_0) \notin \omega^{n+2}\boldsymbol{P}$ then $x_0 \notin A_{n+1}$. Therefore there exists an $g_0 \in \mathfrak{a}$ with $\operatorname{ord}(g_0(x_0)) \leqq \leqq n+1$. Because $\mathfrak{a}$ is a $\boldsymbol{C}[\omega]$-module we can assume $g_0(x_0) - f_n(x_0) \in \omega^{n+2}\boldsymbol{P}$. In virtue of the continuity of the functions g_0 and f_n, there exists to $\varepsilon > 0$ an open neighborhood U_{x_0} of x_0 such that

$$p_{n+1}\big(g_0(x) - f_n(x)\big) < \varepsilon \quad \text{for all} \quad x \in U_{x_0}. \tag{15}$$

We need only finitely many of such neighborhoods $U_1, U_2, \ldots, U_N$, say, to cover the compact space X. For this open covering of X there exists a continuous partition of unity, i.e. a set of N functions $\alpha_k \in \mathscr{C}_{[0,1]}(X)$ such that $\alpha_k(x)=0$ for $x \in X \setminus U_k$, $k=1, 2, \ldots, N$, and $\alpha_1 + \ldots + \alpha_N = \text{const} = 1$ on X (e.g. BOURBAKI [1], Corollary to Proposition IX. 4.4.3.). The functions in $\Gamma\mathfrak{a}$ which replace g_0 in (15) are denoted by g_k, $k=1, 2, \ldots, N$. Then the following argument will prove

$$q_{n+1}\left(f_n - \sum_{k=1}^{N} \alpha_k g_k\right) < \varepsilon. \tag{16}$$

Assume that the point x is contained in $U_1, \ldots, U_r$ but not in $U_{r+1}, \ldots, U_N$, say. Then for this x we get

$$p_{n+1}\left(f_n(x) - \sum_{k=1}^{N} \alpha_k(x) g_k(x)\right) = p_{n+1}\left(\sum_{k=1}^{N} \alpha_k(x) \cdot (f_n(x) - g_k(x)\right) \leqq$$

$$\leqq \sum_{k=1}^{N} \alpha_k(x) \cdot p_{n+1}(f_n(x) - g_k(x)) < \varepsilon,$$

because of $\alpha_1(x) + \ldots + \alpha\ (x) = 1, \alpha_k(x) \geqq 0$. This we can observe for every $x \in X$ and it implies (16) by the very definition (5) of q_{n+1}. On the other side the linear combination $\Sigma \alpha_k g_k$ is an element $a_{n,\varepsilon}$ in the closure $\Gamma\mathfrak{a}$ of $\mathfrak{a}$. Now in virtue of $q_{n+1}(\omega^{n+2}R) = 0$ the seminorm q_{n+1} on R induces the norm q'_{n+1} on $R' = R/\omega^{n+2}R$, a Banach algebra because R' as a Banach space is isometric isomorphic to the direct sum of $n+3$ copies of $\mathscr{C}_{\boldsymbol{C}}(X)$. The closure of the ideal $\mathfrak{a}' = \mathfrak{a} + \omega^{n+2}R$ contains $a'_{n,\varepsilon}$ for every $\varepsilon > 0$. Then (16), considered modulo $\omega^{n+2}R$, implies that f'_n is contained in the closure of $\mathfrak{a}'$ in R'. Therefore there exists an element $b_{n+1} \in \Gamma\mathfrak{a}$ with $q_{n+1}(f_n - b_{n+1}) = 0$. Recall the definition $f_n = f - a_n$ of f_n above. We get $q_{n+1}(f - a_n - b_{n+1}) = q_{n+1}(f_n - b_{n+1}) = 0$ and $a_{n+1} = a_n + b_{n+1} \in \Gamma\mathfrak{a}$. That proves $f \in \Gamma\mathfrak{a}$ and therefore $\chi\varphi\mathfrak{a} \subseteqq \Gamma\mathfrak{a}$. On the other hand $\chi\varphi\mathfrak{a} = \chi(\varphi\mathfrak{a})$, is a closed ideal in R. So we get a characterization of the closure of $\mathfrak{a}$ by

$$\Gamma\mathfrak{a} = \chi\varphi\mathfrak{a} \tag{17}$$

for every ideal $\mathfrak{a}$ in R, and especially the equality

$$\chi\varphi\mathfrak{a} = \mathfrak{a} \tag{18}$$

for every closed ideal $\mathfrak{a}$ in R.

This last result, together with (13), proves the one-to-one correspondence in between the closed ideals in $R = \mathscr{C}_{\boldsymbol{P}}(X)$ and the descending chains $\{A_i; i = -1, 0, 1, 2, \ldots\}$ of closed subsets A_i of X. It is clear that φ is an antiisomorphism of the partial order of the closed ideals in R onto the chains $\{A_i; i\}$, partially ordered component wise:

$$\{A_i; i\} \subseteqq \{B_i; i\} \Leftrightarrow A_i \subseteqq B_i \quad \text{for} \quad i = -1, 0, 1, 2, \ldots. \tag{19}$$

Let us collect these results in the following theorem.

Theorem 1.1. *The* $\boldsymbol{C}$*-algebra* $R = \mathscr{C}_{\boldsymbol{P}}(X)$ *consists of the continuous* $\boldsymbol{P}$*-valued functions on the compact Hausdorff space* X*, where* $\boldsymbol{P}$ *is the complete algebra* $\boldsymbol{C}[[\omega]]$ *of formal power series under its sequence*

$$p_n : \alpha \to |a_0| + |a_i| + \ldots + |a_n| \quad \text{for} \quad \alpha = \sum a_i \omega^i \in \boldsymbol{P} \tag{20}$$

of seminorms $p_n, n = 0, 1, 2, \ldots$. R *is a complete, locally convex* $\boldsymbol{C}$*-algebra with respect to the sequence*

$$q_n f = \sup\{p_n(f(x));\ x \in X\} \quad \textit{for} \quad f \in R \tag{21}$$

of seminorms $q_n, n=0, 1, 2, \ldots$. *If* $V_\Gamma(R)$ *denotes the partial order of closed ideals in* R *and* $V_\Gamma(X)$ *the set of descending chains* $\{A_i; i=-1, 0, 1, 2, \ldots\}$ *of closed subsets* A_i *of* X *beginning with* $A_{-1}=X$ *and partially ordered by component wise inclusion then*

$$\varphi : \mathfrak{a} \to \{\{x\in X;\ a(x)\in\omega^{i+1}\boldsymbol{P}\ \textit{for}\ a\in\mathfrak{a}\}\} \tag{22}$$

is an antiisomorphism of the partial order $V_\Gamma(R)$ *onto* $V_\Gamma(X)$ *with the mapping*

$$\chi = \{A_i;\ i=-1, 0, 1, \ldots\} \to \{f\in R;\ x\in Ai \Rightarrow f(x)\in\omega^{i+1}\boldsymbol{P},\ i=-1, 0, 1, \ldots\} \tag{23}$$

as its inverse. The closure of a (not necessarily closed) ideal $\mathfrak{a}$ *in* R *is*

$$\Gamma\mathfrak{a} = \chi\varphi\mathfrak{a}. \tag{24}$$

If $\{\mathfrak{a}_\lambda; \lambda\in\Lambda\}$ *is a subset of* $V_\Gamma(R)$ *set* $\cup\{\mathfrak{a}_\lambda; \lambda\in\Lambda\}=\Gamma(\sum_{\lambda\in\Lambda}\mathfrak{a}_\lambda)$ *then* $V_\Gamma(R)$ *becomes a complete lattice which is completely lattice antiisomorphic to* $V_\Gamma(X)$ *under* φ *where the operations in* $V_\Gamma(X)$ *are the component wise set intersection and set union respectively of the chains.*

Definition 1.1: A topological ring R is called *topologically arithmetical* if the lattice $V_\Gamma(R)$ of its closed ideals is distributive. Here the operations in $V_\Gamma(R)$ are the set theoretical intersection and the closure of a sum of ideals.

Theorem 1.2. *If* $R=\mathscr{C}_{\boldsymbol{P}}(X)$, *as in theorem* 1, *then* R *is topologically arithmetical.*

Proof. φ is a lattice antiisomorphism of $V_\Gamma(R)$ onto $V_\Gamma(X)$, and $V_\Gamma(X)$ is distributive because the set theoretical union and the intersection of two closed subsets of X are closed.

The distributivity of the lattice $V_\Gamma(R)$, as a lattice of submodules, implies the infinite distributive law

$$\mathfrak{a}\cap\bigcup\{\mathfrak{b}_t;\ t\in T\} = \bigcup\{\mathfrak{a}\cap\mathfrak{b}_t;\ t\in T\}$$

as it is well known. But the dual law is not valid in general, as the following counter example shows:

Take $X=[0, 1]\subseteqq\boldsymbol{R}$, $A=\{1\}$, $B_t=[0, 1-t^{-1}]$ for $t\in\mathbf{N}$. Then $\cup\{B_t; t\in\mathbf{N}\}= =\Gamma(\cup\{B_t;\ t\in\mathbf{N}\})=X$, and therefore $A=A\cap\cup\{B_t;\ t\in\mathbf{N}\}$. But $A\cap B_t=\emptyset$ for any $t\in T$ implies $\cup\{A\cap B_t; t\in T\}=\emptyset$. That means that $V_\Gamma(\mathscr{C}_{\boldsymbol{P}}(X))$ is not D^*-arithmetical in the sense of the author's theory of arithmetical semiperfect rings in Behrens [4] or [5].

2.

If X is a compact Hausdorff space then the closed ideals in $R=\mathscr{C}_{\boldsymbol{P}}(X)$ form a distributive lattice $V_\Gamma(R)$. Under the *Definition*

$$\mathfrak{a}\circ\mathfrak{b} = \Gamma(\mathfrak{a}\cdot\mathfrak{b}) \quad \text{for} \quad \mathfrak{a}, \mathfrak{b}\in V_\Gamma(R) \tag{1}$$

$V_\Gamma(R)$ becomes a lattice ordered semigroup. Its investigation, i.e. the "arithmetic" of R, is facilitated by the fact that for the ring R above the sum $\mathfrak{a}+\mathfrak{b}$ of two closed ideals is closed and their product $\mathfrak{a}\cdot\mathfrak{b}$ is "almost" closed.

This depends on the following Lemma for $\mathscr{C}_C(X)$:

Lemma 2.1. *If A and B are closed subsets of the compact Hausdorff space X and f is a complex valued continuous function on X which is equal to 0 on $A \cap B$, then there exist g and $h \in \mathscr{C}_{\boldsymbol{C}}(X)$ with $f = g + h$ and g being equal to 0 on A and h being equol to 0 on B. If $f \geqq 0$ on X then, in addition, g and h can be taken in $\mathscr{C}_{[0,\infty)}(X)$.*

PROOF. Because $f(x) = 0$ if and only if $\operatorname{Re} f(x)$ and $\operatorname{Im} f(x)$ are both equal to 0 we can assume that f is real valued. Define g' by

$$g'(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ f(x) & \text{if} \quad x \in B. \end{cases}$$

g' is continuous on $A \cup B$ because $f(x) = 0$ for all $x \in A \cap B$. This function g' defined on the closed subset $A \cup B$ of X can be extended to a continuous function g on X (e.g. BOURBAKI [1], Corollary to Theorem 2 (Urysohn) in IX. 4. 2). Then the function $h = f - g \in \mathscr{C}_{\boldsymbol{R}}(X)$ is equal to 0 on B, the function $g \in \mathscr{C}_{\boldsymbol{R}}(X)$ is equal to 0 on A and satisfies $f = g + h$. If $f \in \mathscr{C}_{[0,\infty)}(X)$ then $f = f^+ = g^+ + h^+$, where $g^+ = \max\{g, 0\} \in \mathscr{C}_{[0,\infty)}(X)$.

Theorem 2.2. *The sum $\mathfrak{a} + \mathfrak{b}$ of the closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $R = \mathscr{C}_{\boldsymbol{P}}(X)$, X a compact Hausdorff space, is a closed ideal in R.*

PROOF. Let $f \in \Gamma(\mathfrak{a} + \mathfrak{b})$. We have to show $f \in \mathfrak{a} + \mathfrak{b}$. Now f as an element of R can be written as

$$f = \sum_{i \geqq 0} f_i \omega^i, \quad \text{where} \quad f_i \in \mathscr{C}_{\boldsymbol{C}}(X) \quad \text{for} \quad i = 0, 1, 2, \ldots. \tag{2}$$

By Theorem 1.1 f is contained in $\Gamma(\mathfrak{a} + \mathfrak{b})$ if and only if the following implication holds:

$$x \in A_i \cap B_i \Rightarrow f_i(x) = 0 \quad \text{for} \quad x \in X \quad \text{and all} \quad i = -1, 0, 1, 2, \ldots, \tag{3}$$

where $\{A_i; i\} = \varphi\mathfrak{a}$ and $\{B_i; i\} = \varphi\mathfrak{b}$. By Lemma 2.1 there exist, for every index i, functions g_i and h_i in $\mathscr{C}_{\boldsymbol{C}}(X)$ such that $f_i = g_i + h_i$ and $g_i(A_i) = 0 = h_i(B_i)$. The $\boldsymbol{P}$-valued functions $g = \sum g_i \omega^i$ and $h = \sum h_i \omega_i$ on X are continuous because, for $x_0 \in X$, $n = 1, 2, 3, \ldots$, $\varepsilon > 0$, the inequality

$$p_n\big(g(x) - g(x_0)\big) = \sum_{i=0}^{n} |g_i(x) - g_i(x_0)| < \varepsilon$$

is valid in a convenient neighborhood of x_0; and similarily for h. Then

$$g(x) \in \omega^{i+1}\boldsymbol{P} \quad \text{for} \quad x \in A_i$$

and

$$h(x) \in \omega^{i+1}\boldsymbol{P} \quad \text{for} \quad x \in B_i$$

imply $g \in \mathfrak{a}$, $h \in \mathfrak{b}$ and therefore $f = g + h \in \mathfrak{a} + \mathfrak{b}$.

The $\circ$-product $\mathfrak{a} \circ \mathfrak{b} = \Gamma(\mathfrak{a} \cdot \mathfrak{b})$ of two closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $R = \mathscr{C}_{\boldsymbol{P}}(X)$ is uniquely determined by $\mathfrak{a}$ and $\mathfrak{b}$ and therefore depends from the two chains $\varphi\mathfrak{a}$ and $\varphi\mathfrak{b}$ only, and it must be possible to express $\varphi(\mathfrak{a} \circ \mathfrak{b})$ by the sets A_i in $\varphi\mathfrak{a}$ and B_i in $\varphi\mathfrak{b}$. The corresponding formula becomes more understandable by the following purely set theoretic result:

Lemma. *Let X be a set and denote by $\{A_i;\ -1\leqq i\in\mathbf{Z}\}$ a sequence of subsets A_i of X, beginning with $A_{-1}=X$ and satisfying $A_i\supseteqq A_{i+1}$. These sequences form a lattice ordered monoid M under*

$$\{A_i;\ i\}\cap\{B_i;\ i\}=\{A_i\cap B_i;\ i\}\quad\text{and}\quad\{A_i;\ i\}\cup\{B_i;\ i\}=\{A_i\cup B_i;\ i\}\quad\text{and}$$
$$\{A_i;\ i\}\circ\{B_i;\ i\}=\{C_i;\ i\}\tag{4}$$

where

$$C_i=\bigcup_{j=-1}^{i}(A_j\cap B_{i-1-j}).\tag{5}$$

Its identity is the sequence $\{X;\emptyset,\ldots\}$. M satisfies

$$\{A_i;\ i\}\circ[\{B_i;\ i\}\cap\{C_i;\ i\}]=[\{A_i;\ i\}\circ\{B_i;\ i\}]\cap[\{A_i;\ i\}\circ\{C_i;\ i\}].\tag{6}$$

Proof by verification: The associativity of the $\circ$-operation follows from the fact that C_i in (5) is formed by intersections and unions of subsets of X and that these operations are distributive with another. Similarily we get the compatibility of the $\circ$-operations in M with both lattice operations and the formula (6).

Theorem 2.3. *The lattice antiisomorphism φ of Theorem 1. 1 is a lattice-ordered semigroup-isomorphism of $V_\Gamma(R)$ under $\mathfrak{a}\circ\mathfrak{b}=\Gamma(\mathfrak{a}\cdot\mathfrak{b})$ onto the submonoid $\varphi V_\Gamma(R)$ of M in the lemma, consisting of all descending chains $\{A_i;\ i=-1,0,1,\ldots\}$ of closed subsets A_i of the compact Hausdorff space X which begin with $A_{-1}=X$.*

Proof. Because the union and the intersection of finitely many closed subsets of X are closed and the $\circ$-product in M is defined by (5) we need only to show the equality

$$\varphi(\mathfrak{a}\circ\mathfrak{b})=(\varphi\mathfrak{a})\circ(\varphi\mathfrak{b})\tag{7}$$

for $\mathfrak{a}$ and $\mathfrak{b}$ in $V_\Gamma(R)$. Recall the Definition 1. (6) $\mu(\mathfrak{a},x)=\min\{\operatorname{ord}(f(x)):f\in\mathfrak{a}\}$ for $x\in X$ and $\mathfrak{a}\in V_\Gamma(R)$ and the regula 1, (8),

$$\mu(\mathfrak{a}\circ\mathfrak{b},x)=\mu(\mathfrak{a},x)+\mu(\mathfrak{b},x).\tag{8}$$

Then A_i in $\varphi\mathfrak{a}$ can be expressed by

$$A_i=\{x\in X;\ \mu(\mathfrak{a},x)>i\},\tag{9}$$

and similarily $B_i=\{x\in X;\mu(\mathfrak{b},x)>i\}$ in $\varphi\mathfrak{b}$ and

$$C_i=\{x\in X;\ \mu(\mathfrak{a}\circ\mathfrak{b},x)>i\}\quad\text{in}\quad\varphi(\mathfrak{a}\circ\mathfrak{b}).\tag{10}$$

Then we have the following equivalences for $x\in X$ and $i=-1,0,1,\ldots$

$$x\in C_i\Leftrightarrow\mu(\mathfrak{a}\circ\mathfrak{b},x)>i\Leftrightarrow\mu(\mathfrak{a},x)+\mu(\mathfrak{b},x)>i.\tag{11}$$

$\Leftrightarrow$ There exists an indexpair (i,j) such that

$$i-1=j+k\quad\text{and}\quad x\in A_j\cap B_k$$

$$\Leftrightarrow x\in\bigcup_{j=-1}^{i}(A_j\cap B_{i-1-j})=(\varphi\mathfrak{a})\circ(\varphi\mathfrak{b}),$$

where the last product is formed in M.

The last Theorem provides a managable formula for the calculation of the $\circ$-product $\mathfrak{a}\circ\mathfrak{b}$ in $V_\Gamma(R)$ if $\varphi\mathfrak{a}$ and $\varphi\mathfrak{b}$ are known. That helps in proving the statement that the product $\mathfrak{a}\cdot\mathfrak{b}$ of two closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R is "almost" closed. More precisely, we have the following result:

Theorem 2.4. *The seminorm q_{n-1} of $R=\mathscr{C}_{\mathbf{P}}(X)$, Xa compact Hausdorff space, has the value 0 for all $f\in\omega^n R$ and defines a norm on the factor algebra $R_n'=R|\omega^n R$, Denote by $\mathfrak{a}'$ and $\mathfrak{b}'$ the images of the closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ under the natural epimorphism π of R onto the Banach algebra R_n'. Then $\mathfrak{a}'\cdot\mathfrak{b}'$ is a closed ideal in R^n. Even more is valid: If $f=\sum f_i\omega^i\in\mathfrak{a}\circ\mathfrak{b}=\Gamma(\mathfrak{a}\cdot\mathfrak{b})$ then*

$$f_i\omega^i\in\mathfrak{a}\cdot\mathfrak{b} \quad \textit{for every} \quad i=0,1,2,\ldots \tag{12}$$

and

$$\mathfrak{a}\cdot\mathfrak{b}\subseteqq\mathfrak{a}\circ\mathfrak{b}\subseteqq\bigcap\{\mathfrak{a}\cdot\mathfrak{b}+\omega^n R;\ n=1,2,\ldots\} \tag{13}$$

Proof. It is sufficient to show that the statement (12) is true. Here we can assume $f_n\in\mathscr{C}_{[0,\infty]}(X)$ because for $n=0,1,2,\ldots$, the equality $f_n(x)=0$ is equivalent to $(\operatorname{Re}f_n)(x)=0$, $(\operatorname{Im}f_n)(x)=0$; and $(\operatorname{Re}f_n)(x)=0$ if and only if $(\operatorname{Re}f_n)^+(x)=0$, $(\operatorname{Re}f_n)^-(x)=0$; and similarly for the imaginary part $\operatorname{Im}f_n$ in $f_n=(\operatorname{Re}f_n)^+-(\operatorname{Re}f_n)^-+i[(\operatorname{Im}f_n)^+-(\operatorname{Im}f_n)^-]$. Now $f\in\mathfrak{a}\circ\mathfrak{b}$ implies by the Theorem 2.3 that $f_n(C_n)=0$ where C_n is given by (5).

$n=0$: $C_0=A_0\cup B_0$. Because $f_0\in\mathscr{C}_{[0,\infty]}(X)$ denote by $\sqrt{f_0(x)}$, the non-negative square root of $f_0(x)$ and set $g_0=h_0=\sqrt{f_0}$. Then $f_0=g_0\cdot h_0$ with $g_0(A_0)=0=h_0(B_0)$. Therefore $f_0\omega^0=g_0\omega^0\cdot h_0\omega^0$ with the first factor on the right side being contained in $\mathfrak{a}$ and the second in $\mathfrak{b}$. That settles the case $n=0$.

$n=1$: $C_1=A_1\cup B_1\cup(A_0\cap B_0)$ implies $f_1(A_0\cap B_0)=0$. By Lemma 2.1 there exist g_0 and $h_0\in\mathscr{C}_{[0,\infty]}(X)$ with $\sqrt{f_1}=g_0+h_0$ and $g_0(A_0)=0=h_0(B_0)$. Then the functions g_1 and h_1, defined by

$$g_1(x)=\begin{cases}0 & \text{if} \quad x\in A_1\\ \sqrt{f_1(x)} & \text{if} \quad x\in X\setminus A_1\end{cases}$$

and

$$h_1(x)=\begin{cases}0 & \text{if} \quad x\in B_1\\ \sqrt{f_1(x)} & \text{if} \quad x\in X\setminus B_1,\end{cases}$$

are continuous because $f_1=0$ on A_1 and $f_1=0$ on B_1 respectively. Because $A_1\subseteqq A_0$ and $B_1\subseteqq B_0$ the function $g_0\cdot h_1+g_1\cdot h_0$ is equal to 0 on A_1 and on B_1. The function f_1 equals 0 on $A_1\cup B_1$ in virtue of $f_1(C_1)=0$. But on $(X\setminus A_1)\cap(X\setminus B_1)$ we have the equalities $g_0h_1+g_1h_0=g_0\cdot\sqrt{f_1}+\sqrt{f_1}\cdot h_0=(g_0+h_0)\cdot\sqrt{f_1}=\sqrt{f_1}\cdot\sqrt{f_1}=f_1$. That proves

$$f_1\omega=(g_0+g_1\omega)\cdot(h_0+h_1\omega)-g_0\omega^0\cdot h_0\omega^0-g_1\omega\cdot h_1\omega\in\mathfrak{a}\cdot\mathfrak{b}$$

and settles the case $n=1$.

$n \geqq 2$: Because $A_{-1}=X=B_{-1}$ we have

$$C_n = B_n \cup (A_0 \cap B_{n-1}) \cup (A_1 \cap B_{n-2}) \cup \ldots \cup A_n = [\ ]_1 \cap [\ ]_2,$$

where

$$[\ ]_1 = A_n \cup B_n \cup (A_0 \cap B_0) \tag{14}$$

and

$$[\ ]_2 = A_{n-1} \cup B_{n-1} \cup (A_1 \cap B_{n-2}) \cup \ldots \cup (A_{n-2} \cap B_1) \tag{15}$$

are closed subsets of X. This representation of C_n as the intersection of $[\ ]_1$ and $[\ ]_2$ implies for the function f_n a representation $f_n = u + v$ where $u, v \in \mathscr{C}_{[0,\infty]}(X)$ and $u=0$ on $[\ ]_1$ and $v=0$ on $[\ ]_2$ by Lemma 2.1, observing $f_n(C_n)=0$. If we replace A_1 by A_n and B_1 by B_n in the discussion of the case $n=1$ above we get four functions g_0, h_0, g_n, h_n in $\mathscr{C}_{[0,\infty]}(X)$ satisfying $g_0(A_0)=0=h_0(B_0)$ and $g_n(A_n)=0=h_n(B_n)$ and $g_0 \cdot h_n + h_0 \cdot g_n = u$. These equalities prove that $g_0 + g_n\omega^n$, $g_0\omega^0$ and $g_n\omega^n$ are functions in the ideal $\mathfrak{a}$ and that $h_0 + h_n\omega^n$, $h_0\omega^0$ and $h_n\omega^n$ are contained in the ideal $\mathfrak{b}$. Therefore the function $u\omega^n = (g_0 + g_n\omega^n)(h_0 + h_n\omega^n) - g_0\omega^0 \cdot h_0\omega^0 - g_n\omega^n \cdot h_n\omega^n$ is contained in $\mathfrak{a} \cdot \mathfrak{b}$. Similarily we deal with the second sum and v in $f_n = u + v$, recalling v equals 0 on $[\ ]_2$ and using an induction argument. More precisely, we omit A_0 and B_0 in the chains $\varphi\mathfrak{a}$ and $\varphi\mathfrak{b}$ respectively and get the chains $\{X, A_1, A_2, \ldots\} = \varphi\mathfrak{a}_1$ and $\{X, B_1, B_2, \ldots\} = \varphi\mathfrak{b}_1$ of two closed ideals $\mathfrak{a}_1$ and $\mathfrak{b}_1$ respectively. An element $f \in R$ belongs to $\mathfrak{a}_1$ if and only if $\omega f \in \mathfrak{a}$, in virtue of the definitions for φ and χ in Theorem 2.1, and similarily for $\mathfrak{b}_1$. Then by the induction assumption, that the satement (12) is true for $0 \leqq i \leqq n-1$ and all ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $V_\Gamma(R)$, and by considering the definition (15) of $[\ ]_2$ we get $v\omega^{n-2} \in \mathfrak{a}_1 \cdot \mathfrak{b}_1$ and therefore $v\omega^n \in \mathfrak{a} \cdot \mathfrak{b}$. Because the cases $n=0$ and $n=1$ are settled already, that proves $f_n\omega^n = u\omega^n + v\omega^n \in \mathfrak{a} \cdot \mathfrak{b}$.

Corollary 2.5: The n-th power $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$ is closed and $\varphi(\mathfrak{m}^n) = \{X, x, \ldots, x, \emptyset, \ldots\}$ is valid, where the chain contains n copies of the point $x \in X$. The powers $\mathfrak{m}^n$ form a strictly descending chain.

Proof. $\mathfrak{m}$ as a maximal element in $V_\Gamma(R)$ has $\varphi\mathfrak{m} = \{X, x, \emptyset, \ldots\}$ as its associated chain with an uniquely determined point $x \in X$. On the other hand $\varphi(\omega R) = \{X, X, \emptyset, \ldots\}$ shows $\omega R \subseteqq \mathfrak{m}$ and therefore $\omega^n R \subseteqq \mathfrak{m}^n$.

Corollary 2.6: The (Jacobson) radical of R is $\mathfrak{j} = \omega R = \chi(\{X, X, \ldots\}) = \cap \{\mathfrak{m}; \mathfrak{m} \in \mathfrak{X}\}$ where $\mathfrak{X}$ is the set of the maximal ideals in R. $\mathfrak{j}^n = \cap\{\mathfrak{m}^n; \mathfrak{m} \in \mathfrak{X}\}$ if $n = 1, 2, 3, \ldots$.

Remark: $R|\mathfrak{j}$ is isometric isomorphic to the commutative B^*-algebra $\mathscr{C}_{\boldsymbol{C}}(X)$.

Proof *of the Corollary:* $\mathfrak{j}$ is the intersection of the maximal ideals in the commutative algebra R with identity element by definition.

Corollary 2.7: If $\mathfrak{m}$ is a maximal ideal in R then

$$\mathfrak{n}_\mathfrak{m} = \cap\{\mathfrak{m}^n;\ n = 1, 2, 3, \ldots\} \tag{16}$$

is a closed ideal in R and

$$\cap\{\mathfrak{n}_{\mathfrak{m}};\ \mathfrak{m}\in\mathfrak{X}\} = (0). \tag{17}$$

The ideal $\mathfrak{n}_{\mathfrak{m}}$ replaces the maximal ideal $\mathfrak{m}$ in the Gelfand—Mazur setting in $\mathscr{C}_{\boldsymbol{C}}(X)$:

Theorem 2.8. *Let $R=\mathscr{C}_{\boldsymbol{P}}(X)$, X a compact Hausdorff space, and $\mathfrak{m}_0=\chi(\{X, x_0, \emptyset, \ldots\})$ a maximal ideal in R. Then*

$$\sigma_0 : \tilde{f} = f+\mathfrak{n}_{\mathfrak{m}_0} \to f(x_0) \quad \text{for} \quad \tilde{f}\in\tilde{R} \tag{18}$$

is an isomorphism of $R|\mathfrak{n}_{\mathfrak{m}_0}$ onto $\boldsymbol{P}$ as an $\boldsymbol{C}$-algebra. The n-th seminorm q_n of R induces on $\tilde{R}=R|\mathfrak{n}_{\mathfrak{m}}$ the seminorm $\tilde{q}_n:\tilde{f}\to\inf\{q_n(f+k);\ k\in\mathfrak{n}_{\mathfrak{m}}\}$ for $\tilde{f}\in\tilde{R}$. Then

$$p_n(\sigma_0\tilde{f}) = \tilde{q}_n(\tilde{f}) \quad \textit{for} \quad \tilde{f}\in\tilde{R} \quad \textit{and all} \quad n = 1, 0, 2, \ldots. \tag{19}$$

In this sense σ_0 is an "isometric" isomorphism of $R|\mathfrak{n}_{\mathfrak{m}}$ onto $\boldsymbol{P}$.

PROOF. In virtue of $\varphi\mathfrak{n}_{\mathfrak{m}_0}=\{X, x_0, x_0, x_0, \ldots\}$ a function $g\in R$ belongs to $\mathfrak{n}_{\mathfrak{m}_0}$ if and only if $g(x_0)=0$, the zero element in $\boldsymbol{P}$. Therefore

$$\eta_{\mathfrak{m}_0} : f \to f(x_0) \quad \text{for} \quad f\in R$$

is an $\boldsymbol{C}$-algebra epimorphism of R to $\boldsymbol{P}$ with ker $\eta_{\mathfrak{m}_0}=\mathfrak{n}_{\mathfrak{m}_0}$ and σ_0 is an algebraic isomorphism. Because $\mathfrak{n}_{\mathfrak{m}}$ is a closed ideal in R the seminorm $\tilde{q}_n$ induced by q_n is given by

$$\tilde{q}_n(\tilde{f}) = \inf\{q_n(f+k);\ k\in\mathfrak{n}_{\mathfrak{m}}\},$$

where

$$q_n(f+k) = \sup\{p_n(f(x)+k(x));\ x\in X\}.$$

Now $f=f(x_0)e+g$, where e is the identity element in R and $g\in\mathfrak{n}_{\mathfrak{m}}$, implies $q_n(f+k)=\sup\{p_n(f(x_0)+g(x)+k(x));\ x\in X\}$. Because g and k are continuous on the compact Hausdorff space X there exists an $y\in X$ with $q_n(f+k)=p_n(f(x_0)+g(y)+k(y))\geqq p_n(f(x_0))$. On the other hand we get $\tilde{q}_n(\tilde{f})=\inf p_n(f(x_0)+g(y)+k(y));\ k\in\mathfrak{n}_{\mathfrak{m}}\}\leqq p_n(f(x_0))$ because with k the sum $g+k$ is running through $\mathfrak{n}_{\mathfrak{m}}$. That proves (19).

Corollary 2.9: The sequence $\{\mathfrak{m}^n;\ n\in\overline{N}^0\}$ of closed ideals in R, joining R and $\mathfrak{n}_{\mathfrak{m}}$, is a maximal chain of $\boldsymbol{C}$-subspaces of he $\boldsymbol{C}$-linear space $R/\mathfrak{n}_{\mathfrak{m}}$.

PROOF. $\mathfrak{m}^n|\mathfrak{m}^{n+1}$ is an $R|\mathfrak{m}$-module and $R|\mathfrak{m}\cong\boldsymbol{C}$.

3.

In the classical number theory it is shown that every ideal is the intersection of finitely many powers of maximal ideals and then that this intersection can be replaced by the product of these finitely many powers. In $R=\mathscr{C}_{\boldsymbol{P}}(X)$, X a compact Hausdorff space, the ideals $\mathfrak{a}$ possess infinitely many components, $\mathfrak{m}^{\alpha(\mathfrak{m})}$ in general. That inhibits a product representation.

Take $\mathfrak{a}$ in $V_\Gamma(R)$, the lattice ordered semigroup of the closed ideals in $R = \mathscr{C}_{\boldsymbol{P}}(X)$, as in Theorem 2.3. Then the points x in the subset A_i of X correspond to the maximal ideals $\mathfrak{m}_x = \chi(\{X, x, \emptyset, \ldots\})$ in R and, by Corollary 2.5, $\mathfrak{m}_x^\alpha = \chi(\{X, x, \ldots, x, \emptyset, \ldots\})$ with α copies of the point $x \in X$. Considering 1, (22), and 1, (6), we have

$$(1)\qquad A_i = \{x \in X;\ \mu(\mathfrak{a}, x) > i\} \quad \text{for} \quad i = -1, 0, 1, \ldots \quad \text{and} \quad \mathfrak{a} \in V_\Gamma(R).$$

So we get by the theorems 1.1, 2.3, 2.4 and the Corollaries 2.5, 2.7 for the closed ideals in R the representation

$$(2)\qquad \mathfrak{a} = \bigcap \{\mathfrak{m}^{\mu(\mathfrak{a}, \mathfrak{m})};\ \mathfrak{m} \in \mathfrak{X}\}$$

where $\mathfrak{X}$ is the set of the maximal ideals in X and

$$(3)\qquad \mu(\mathfrak{a}, \mathfrak{m}) = \mu(\mathfrak{a}, x) \quad \text{for} \quad \mathfrak{m} = \chi(X, x, \emptyset, \ldots).$$

Here we set $\mathfrak{m}^0 = R$ and $\mathfrak{m}^\infty = \mathfrak{n}_\mathfrak{m}$. The range of the functions

$$(4)\qquad \mu(\mathfrak{a}, -) : \mathfrak{m} \to \mu(\mathfrak{a}, \mathfrak{m}) \quad \text{for} \quad \mathfrak{m} \in \mathfrak{X}$$

consists of non-negative rational integers and ∞, i.e. it is contained in $\overline{\boldsymbol{N}}^0$. These functions are not quite arbitrary because the subsets A_i of X in the chain $\varphi\mathfrak{a}$ are closed. In fact, if we begin with an arbitrary $\overline{\boldsymbol{N}}^0$-valued function α on X then

$$(5)\qquad \mathfrak{a} = \bigcap \{\mathfrak{m}_x^{\alpha(x)};\ x \in X\}$$

is as an intersection of closed ideals a closed ideal in R and therefore

$$(6)\qquad \mathfrak{a} = \bigcap \{\mathfrak{m}^{\mu(\mathfrak{a}, x)};\ x \in X\}$$

with $\mu(\mathfrak{a}, x) = \mu(\mathfrak{a}, \mathfrak{m}_x)$ for $x \in X$ also. We derive $\mu(\mathfrak{a}, -)$ from α by the upper semi-continuous regularization of the $\overline{\boldsymbol{N}}^0$-valued function α (Bourbaki [1], IV.6.2) using the following argument: For $\mathfrak{b} \in V_\Gamma(R)$ define $A_i(\mathfrak{b})$ by

$$\varphi\mathfrak{b} = \{A_i(\mathfrak{b});\ i = -1, 0, 1, \ldots\}.$$

Then for $\mathfrak{a}$, given by (5), we calculate by Theorem 1.1

$$A_i(\mathfrak{a}) = \Gamma(\cup \{A_i(\mathfrak{m}_x^{\alpha(x)});\ x \in X)$$

where by Corollary 2.5

$$A_i(\mathfrak{m}_x^{\alpha(x)}) = \begin{cases} x & \text{if} \quad -1 \leqq i < \alpha(x) \\ \emptyset & \text{if} \quad i \geqq \alpha(x). \end{cases}$$

That shows

$$A_i(\mathfrak{a}) = \Gamma\{x \in X;\ \alpha(x) > i\}.$$

Therefore a point $x_0 \in X$ belongs to $A_i(\mathfrak{a})$ if and only if every neighborhood U of x_0 contains a point x_U with $\alpha(x_U) > i$, with other word, if and only if $\sup\{\alpha(y);\ y \in U\} > i$ for all neighborhoods U of x_0. That is equivalent to

$$(7)\qquad i < \inf\big\{\sup\{\alpha(y);\ y \in U\};\ x_0 \in U,\ U \text{ open}\big\}.$$

The right side of (7) is called the limit superior of α for $y \to x$. Recalling $A_i(\mathfrak{a}) = \{x \in X; \mu(\mathfrak{a}, x) > i\}$ we get

$$\mu(\mathfrak{a}, x_0) = \limsup_{y \to x_0} \alpha(y), \quad \text{for} \quad x \in X, \tag{8}$$

i.e. the function $\mu(\mathfrak{a}, -)$ is the upper semi-continuous regularization of the $\bar{N}^0$-valued function α on X. The function α is called upper semicontinuous if α equals its upper semicontinuous regularization. The upper continuous functions on X with values in $\bar{N}^0$ form a lattice ordered monoid $\mathfrak{S}$ under the following operations for $\alpha, \beta \in \mathfrak{S}$:

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x)$$

$$(\alpha \cap \beta)(x) = \sup\{\alpha(x), \beta(x)\}, \ (\alpha \cup \beta)(x) = \inf\{\alpha(x), \beta(x)\}.$$

(BIRKHOFF [1], § XIII. 4).

These considerations together with formula 1, (8) for $\mu(\mathfrak{a} \circ \mathfrak{b}, x)$ and Theorem 1.1 describe the arithmetic of R, i.e. the lattice ordered semigroup of its closed idelas. They are collected in the following *Main Theorem:*

Theorem 3.1. *Let X be a compact Hausdorff space and set $R = \mathscr{C}_{\boldsymbol{P}}(X)$ where $\boldsymbol{P} = \boldsymbol{C}[[\omega]]$, ω an indeterminate. The maximal ideals $\mathfrak{m}$ in R correspond one-to-one to the points in X under*

$$\mathfrak{m} = \mathfrak{m}_x = \{f \in R; f(x) \in \omega \boldsymbol{P}\}.$$

The powers $\mathfrak{m}^n, n = 1, 2, \ldots,$ of $\mathfrak{m}$ are closed and $\mathfrak{n}_{\mathfrak{m}} = \bigcap \{\mathfrak{m}^n\}$ is closed also. The closed ideals in R form a lattice ordered semigroup $V_\Gamma(R)$ under the following operations $\mathfrak{a} \circ \mathfrak{b} = \Gamma(\mathfrak{a} \cdot \mathfrak{b})$, $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$. Let $\mathfrak{S}$ be the lattice ordered semigroup of the upper semicontinuous $\bar{\mathrm{N}}^0$-valued functions on X with the operations

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x) \tag{9}$$

and

$$(\alpha \cap \beta)(x) = \sup\{\alpha(x), \beta(x)\} \quad \text{and} \quad (\alpha \cap \beta)(x) = \inf\{\alpha(x), \beta(x)\}. \tag{10}$$

Set $\mathfrak{m}^0 = R$ and $\mathfrak{m}^\infty = \mathfrak{n}_{\mathfrak{m}}$. Then the mapping

$$\psi : \alpha \to \bigcap \{\mathfrak{m}_x^{\alpha(x)}, x \in X\} \tag{11}$$

for $\alpha \in \mathfrak{S}$ is an isomorphism of lattice ordered semigroups from $\mathfrak{S}$ onto $V_\Gamma(R)$. The isomorphism ψ is complete if $\bigcup\{\mathfrak{a}_i; i \in I\}$ is defined as $\Gamma\big(\sum_i \mathfrak{a}_i\big)$ for a subset $\{\mathfrak{a}_i; i \in I\}$ of $V_\Gamma(R)$.

4.

If R is a commutative semiperfect ring with identity element and if its lattice $V(R)$ is distributive then every ideal in R is principal by Behrens [4], IX. 2 and Theorem 5 in IX. 1. So we ask whether every closed ideal $\mathfrak{a}$ in the ring $R = \mathscr{C}_{\boldsymbol{P}}(X)$, is the closure of a principal ideal. We will show that this is the case if and only if the compact Hausdorff space X is perfectly normal. A normal topological space X is called perfectly normal (BOURBAKI [1], Exercise 7 to IX. 4) if every closed subset A of X is a countable intersection of open subsets of X or, equivalently, if and only if there exists a function $f \in \mathscr{C}_{\boldsymbol{R}}(X)$ such that $f(x)=0$ for $x \in A$ and $f(x) \neq 0$ for $x \notin A$.

Theorem 4.1. *Let X be a compact Hausdorff space. The ring $R=\mathscr{C}_{\boldsymbol{R}}(X)$ is a topological principal ideal ring, i.e. every closed ideal $\mathfrak{a}$ in R is the closure $\Gamma(Rf)$ of a principal ideal Rf in R, if and only if X is perfectly normal.*

PROOF. 1) Let $\mathfrak{a} \in V_\Gamma(R)$ and $\varphi\mathfrak{a}=\{A_i; i=-1, 0, 1, \ldots\}$ be its associated descending chain of closed subsets A_i of X. If X is perfectly normal there exist $f_i \in \mathscr{C}_{[0,1]}(X)$ with $A_i=f_i^{-1}(0)$ for i. The function $f=\sum_{i \geqq 0} f_i \omega^i \in \mathscr{C}_{\boldsymbol{P}}(X)$ generates an ideal Rf. Its closure $\Gamma(Rf)$ has as its associated chain $\varphi(\Gamma(Rf))$ the chain $\varphi\mathfrak{a}$ because a point $x \in X$ satisfies $g(x) \in \omega^{n+1}\boldsymbol{P}$, for all $g \in \Gamma(Rf)$ if and only if $f(x) \in \omega^{n+1}\boldsymbol{P}$, in virtue of the continuity of the functions in R. That proves $\mathfrak{a}=\Gamma(Rf)$, — Conversely, take a closed subset A of the compact Hausdorff space X and assume that the ideal $(0) \neq \mathfrak{a}=\chi(\{X, A, \emptyset, \ldots\})$ in $V_\Gamma(R)$ is topologically principal, $\mathfrak{a}=\Gamma(Rf)$, say; $f=f_0+f_1\omega+\ldots$. Then

$$g = \|f_0\|^{-1} \cdot |f_0|$$

is a function in $\mathscr{C}_{[0,1]}(X)$ such that $g^{-1}(0)=A$.

The ring $R=\mathscr{C}_P(X)$ contains very few closed principal ideals:

Theorem 4.2. *Let X be a compact Hausdorff space. Then the powers $\omega^n R$ of the radical ωR of $R=\mathscr{C}_{\boldsymbol{P}}(X)$ are the only closed ideals in R which are principal ideals.*

PROOF. By Corollary 2.6 the ideals $\omega^n R$ are closed and generated by $\omega^n e$, e the function constantly equal to 1 on X. Assume that $\mathfrak{a} \in V_\Gamma(R)$ is different from all powers $\omega^n R$. Then there exists a smallest index m such that $A_{m-1}=X$ but $X \supset A_m \supset \emptyset$, where $\varphi\mathfrak{a}=\{A_i; i=-1, 0, 1, \ldots\}$. Assume $\mathfrak{a}=Rf$ already. Then $f=f_0+f_1\omega+\ldots$ with $f_0=\ldots=f_{m-1}=\text{const.}=0$ and $f_m(x)=0$ if and only if $x \in A_m$. the function $\sqrt{|f_m|} \cdot \omega^m$ belongs to $\chi\varphi\mathfrak{a}=\mathfrak{a}=Rf$. That implies the existence of a function $g=g_0+g_1\omega+\ldots$ in $\mathscr{C}_{\boldsymbol{P}}(X)$ satisfying

$$\sqrt{|f_m|} \cdot \omega^m = g \cdot f.$$

Then $\sqrt{|f_m|}=f_m \cdot g_0$, and therefore

$$\sqrt{|f_m(x)|} \cdot |g_0(x)| = \begin{cases} 0 & \text{if } x \in A_m, \\ 1 & \text{if } x \notin A_m \end{cases}$$

is valid in contradiction to the continuity of $|f_m| \cdot |g_0|$ on X.

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