On products of full linear rings

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To the memory of Andor Kertész

Recently GOLDMAN [2] showed that a ring (with unity element) is a product of full linear rings (of left vector spaces over division rings) if and only if R is Hausdorff and complete in the right *intrinsic* topology, which has a fundamental system of open neighborhoods of zero consisting of all right ideals K such that R/K is a completely reducible projective right R-module.

Ring theoretic characterizations of products of full linear rings were given by CHASE and FAITH [1]. It is the purpose of this note to show that both characterizations follow easily from Jacobson's density theorem as formulated in [6] and some easy results on bicommutators proved here. It is also shown that the intrinsic topology on R coincides with the $\bar{I}$-adic topology. In general, the *I-adic* topology on a right R-module A has a fundamental system of open neighborhoods of zero consisting of all kernels of homomorphisms $A \to I^n$, where n is any natural number.

Given right R-modules I and A, we adopt the notation from [6]:

$$S_I(A) = \mathrm{Hom}_{\mathrm{End}\,(I)}(\mathrm{Hom}_R(A, I), I)$$

where I is regarded as a left module over its ring of endomorphisms End (I). In particular,

$$S_I(R) \cong \mathrm{Hom}_{\mathrm{End}\,(I)}(I, I) = \mathrm{Bic}\,(I)$$

is the *bicommutator* of I, which is regarded as a right $S_I(R)$-module. We recall the following from [6, Remark 5, 3]:

Lemma 0. *If I is a completely reducible right R-module, the canonical module homomorphism $A \to S_I(A)$ is the Hausdorff completion of A in the I-adic topology. In particular, the canonical ring homomorphism $R \to$ Bic (I) is the Hausdorff completion of R.*

Given a right R-module V and a set X, we write

$$XV = \sum_{\alpha \in X}^{\oplus} V_\alpha.$$

where $V_\alpha = V$ for all $\alpha \in X$. We denote by $p_\alpha : XV \to V$ and $k_\alpha : V \to XV$ the canonical projection and injection corresponding to $\alpha \in X$.

Lemma 1. *Let V and A be right R-modules, X any set. If X is finite or A is finitely generated, then*

$$S_{XV}(A) \cong S_V(A).$$

In particular,

$$\mathrm{Bic}\,(XV) \cong \mathrm{Bic}\,(V).$$

PROOF. Let $\beta \in X$ be fixed. For any $s \in S_{XV}(A)$ any $g \in \mathrm{Hom}_R(A, V)$ we write

$$(g)s^* = p_\beta((k_\beta g)s).$$

(According to the usual convention, homomorphisms of left modules act from the right.) Then * is an R-homomorphism $S_{XV}(A) \to S_V(A)$.

For any $t \in S_V(A)$ and $f \in \mathrm{Hom}_R(A, XV)$, we write

$$t^+(f) = \sum_{\alpha \in X} k_\alpha((p_\alpha f)t).$$

For this formula to make sense, we require that $p_\alpha f = 0$ for all but a finite number of α. This will be so if X is finite or if A is finitely generated. A straight forward calculation shows that $^+$ is the inverse of *.

Lemma 2. *Let A and I be right R-modules and $I = \sum_{\alpha \in X}^{\oplus} I_\alpha$, where $Hom_R(I_\alpha, I_\beta) = 0$ if $\alpha \neq \beta$. If X is finite or A is finitely generated, then*

$$S_I(A) \cong \prod_{a \in X} S_{I_\alpha}(A).$$

In particular,

$$\mathrm{Bic}\,(I) \cong \prod_{\alpha \in X} \mathrm{Bic}\,(I_\alpha).$$

PROOF. We denote by $p_\alpha: I \to I_\alpha$ and $k_\alpha: I_\alpha \to I$ the canonical projection and injection corresponding to $\alpha \in X$.

To each $s \in S_I(A)$ we assign $s^* \in \prod_{\alpha \in X} S_{I_\alpha}(A)$ by defining, for any $f_\alpha \in \mathrm{Hom}_R(A, I_\alpha)$, the component $s^* \in S_{I_\alpha}(A)$ by

$$(f_\alpha)s_\alpha^* = p_\alpha((k_\alpha f_\alpha)s).$$

To each $t \in \sum_{\alpha \in X} S_{I_\alpha}(A)$, with components $t_\alpha \in S_{I_\alpha}(A)$, we assign $t^+ \in S_I(A)$ by defining, for any $f \in \mathrm{Hom}_R(A, I)$,

$$(f)t^+ = \sum_{\alpha \in X} k_\alpha((p_\alpha f)t_\alpha).$$

This formula makes sense if $p_\alpha f = 0$ for all but a finite number of $\alpha \in X$, and this is so if X is finite or A is finitely generated.

Under the assumption that $\mathrm{Hom}_R(I_\alpha, I_\beta) = 0$ for $\alpha \neq \beta$, one easily computes that $^+$ is the inverse of *.

We recall that the left socle and right socle of a semiprime ring coincide [5, § 3.4, Proposition 4].

Lemma 3. *Let R be a semiprime ring and suppose that its socle S is essential as a left ideal. Then S is also essential as a right ideal, and the bicommutator of the right R-module S_R is an essential extension of the left R-module ${}_RR$.*

PROOF. Suppose $0 \neq r \in R$. Since ${}_RS$ is essential, Rr contains a nonzero idempotent $e = r'r$. Then $r'rer'r = e^3 = e \neq 0$, hence $0 \neq rer' \in S$. Therefore, S_R is essential.

Since R is semiprime, the right annihilator of S is zero. It follows that the canonical mapping $R \to \mathrm{Bic}(S_R)$ is a monomorphism. Let $0 \neq q \in \mathrm{Bic}(S_R)$, then we can find an idempotent e such that $0 \neq eq \in S \subseteq R$. Therefore, $\mathrm{Bic}(S_R)$ is an essential extension of ${}_RR$.

Lemma 4. *For a semiprime ring with socle S, the S-adic topology coincides with the intrinsic topology.*

PROOF. Let K be a fundamental open neighborhood of zero in the S-adic topology of R_R, that is, the kernel of a homomorphism $f: R \to S^n$. Then $R/K \cong \cong \mathrm{im} f \subseteq S^n \subseteq S$ is a direct sum of minimal right ideals of the form eR, where $e^2 = e$, hence completely reducible and projective. Therefore, K is a fundamental open neighborhood of zero in the intrinsic topology.

Conversely, suppose R/K is completely reducible and projective. Then $K = eR$, for some idempotent $e \in R$, and $R/K \cong (1-e)R \subseteq S$. Therefore, K is the kernel of a homomorphism $R \to S$.

For completeness, we shall also record the following wellknown facts.

Lemma 5. *Let ${}_DV$ be a left vector space over a division ring D and $R = \mathrm{Hom}_D(V, V)$. Then R is von Neumann regular, self-injective and its socle is essential as a left ideal.*

PROOF. It is well-known that R is regular [see e.g. 5, § 4.4, Proposition 1]. Thus V_R is flat, and since ${}_DV$ is injective, it follows that ${}_RR$ is injective [5, § 5.3, Proposition 3]. To see that the socle S is essential as a left ideal, let $0 \neq r \in R$, then $vr \neq 0$ for some $v \in V$. Let $V = Dv \oplus W$ and define $e \in R$ by $ve = 1$, $We = 0$. Then $ver = vr \neq 0$ hence $er \neq 0$. But $er \in Rer \subseteq S$, since Re is a minimal left ideal of R, as is easily checked.

Theorem. *The following properties of a ring R are equivalent:*

(1) *R is Hausdorff and complete in the right intrinsic topology.*

(2) *R is Hausdorff and complete in the S-adic topology, where S is the socle of R regarded as a right R-module.*

(3) *R is Hausdorff and complete in the I-adic topology, where I is some completely reducible right R-module.*

(4) *R is isomorphic to the bicommutator of some completely reducible right R-module.*

(5) *R is a direct product of full linear rings of left vector spaces.*

(6) *R is von Neumann regular, left self-injective and its socle is essential as a left ideal.*

(7) *R is semiprime left self-injective and its socle is essential as a left ideal.*

(8) *The canonical homomorphism $R \to \mathrm{Bic}(S)$, where S is the right socle of R, is an isomorphism.*

PROOF. We shall establish the following implications:

$(1)\Leftrightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)\Rightarrow(7)\Rightarrow(8)\Rightarrow(2)$.

$(1)\Leftrightarrow(2)$ by Lemma 4, because R is semiprime under either hypothesis.

$(2)\Rightarrow(3)$ is trivial.

$(3)\Rightarrow(4)$ follows from Lemma 0.

$(4)\Rightarrow(5)$: Suppose R is isomorphic to the bicommutator of a completely reducible module I. We may write $I=\sum_{\alpha\in X}^{\oplus} I_\alpha$, where the I_α are the homogeneous components of I, and $I_\alpha=X_\alpha V_\alpha$, X_α being a set and V_α a fixed minimal submodule of I_α. By lemmas (2) and (1), we have

$$R \cong \mathrm{Bic}(I) \cong \prod_{\alpha\in X} \mathrm{Bic}(I_\alpha) \cong \prod_{\alpha\in X} \mathrm{Bic}(V_\alpha) = \prod \mathrm{End}_{D_\alpha}(V_\alpha),$$

where $D_\alpha=\mathrm{End}_R(I_\alpha)$.

$(5)\Rightarrow(6)$: Each $R_\alpha=\mathrm{End}_{D_\alpha}(V_\alpha)$ is regular, self-injective and its scole is essential as a left ideal, by Lemma 5. Therefore, R is left self-injective [5, § 4.3, Proposition 9] and von Neumann regular (obvious), and it is easily seen that the socle of R is left essential.

$(6)\Rightarrow(7)$ is trivial.

$(7)\Rightarrow(8)$: By Lemma 3, the bicommutator of S_R is an essential extension of $_RR$. Since $_RR$ is injective, $R\cong\mathrm{Bic}(S_R)$ canonically.

$(8)\Rightarrow(2)$ follows from Lemma 0.

COROLLARY 1. *The complete ring of left quotients Q of a ring R is a product of full linear rings of left vector spaces if and only if the following conditions hold:*

(1) *The left singular ideal Z of R is zero.*

(2) *Every nonzero left ideal of R contains a nonzero uniform left ideal.*

PROOF. Assume the conditions. By (1), Q is von Neumann regular and left self-injective [5, § 4.5]. It will follow from the theorem that Q is a full linear ring if we show that its socle is left essential. So, let L be a nonzero left ideal of Q, we want to show that L contains a minimal left ideal. Now $L\cap R\neq 0$; hence, by (2), it contains a uniform left ideal $U\neq 0$ of R, and $QU\subseteqq L$. We claim that QU is a minimal left ideal of Q. Since Q is regular, it suffices to check that QU is uniform. (Let $0\neq a, b\in QU$. Since Q is a left ring of quotients of R, we can find $r, s\in R$ such that $0\neq ra, sb\in U$. Therefore $Rra\cap Rrb\neq 0$, hence $Qa\cap Qb\neq 0$. Thus QU is uniform.)

Conversely, suppose Q is a product of full linear rings. Then, by the theorem, Q is von Neumann regular, left self-injective, and its socle S is left essential. By By [5, § 4.5], $Z=0$. Let L be a nonzero left ideal of R, then $QL\cap S\neq 0$, and so QL contains a minimal left ideal V of Q. Since Q is a ring of left quotients of R, $R\cap V$ is a nonzero uniform left ideal of R. (For, if $0\neq a, b\in R\cap V$, we can find $q, q'\in Q$ such that $0\neq qa=q'b\in V$, then find $r\in R$ such that $rq\in R$, $rq'\in R$ and $0\neq rqa=rq'b\in V$.) Let $0\neq v\in V\subseteqq QL$, we can find $r\in R$ such that $0\neq rv\in L$. Hence Rrv will be a nonzero left ideal contained in $V\cap L$, hence a uniform left ideal contained in L.

The sufficiency of conditions (1) and (2) is due to JOHNSON [4, Theorem 3.1], see also [3, Theorem 3.8].

COROLLARY 2. *Let R be a semiprime ring whose socle is an essential left ideal. Then its complete ring of left quotients is a product of full linear rings of left vector spaces.*

PROOF. In view of Corollary 1, we need only verify the two conditions.

(1) Since every essential left ideal contains the socle S, $(S\cap Z)^2 \subseteqq SZ=0$, hence $S\cap Z=0$, and so $Z=0$.

(2) Every nonzero left ideal has nonzero intersection with S, hence contains a minimal left ideal.

This corollary is a generalization of a theorem by UTUMI [5, § 4.3, Proposition 7] also found in [1]. One could use it for obtaining yet another proof of Goldie's theorem on semiprime rings.

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