Subproducts and subdirect products

By F. LOONSTRA. (Delft)

To the memory of Prof. A. Kertész

§ 1. Introduction

In the following R is a *commutative* ring $(1\in R)$, $\{M_i\}_{i\in I}$ a (non-empty) set of R-modules $M_i\neq 0$, F an R-module, and $\alpha_i: M_i\to F (i\in I)$ a set of R-epimorphism. We consider the set $M\subset \prod_{i\in I} M_i$ of those elements $m=(m_i)_{i\in I}\in \prod_{i\in I} M_i$ such that the following relations are satisfied:

$$(1)\qquad \sum_{i\in I} r_{ji}\alpha_i(m_i)=0 \quad (j\in J),$$

where I and J are index sets, and where every sum in (1) contains only a finite number of terms $\neq 0$. M is a submodule of $\prod_{i\in I} M_i$, called a *subproduct* M of the M_i, denoted by

$$(2)\qquad M=\Big\{M_i;\ \alpha_i;\ F;\ \sum_{i\in I} r_{ji}\alpha_i(m_i)=0;\ j\in J\Big\}.$$

A well known example of (2) is the *special subdirect product*

$$M=\bigotimes_{i\in I} M_i(\alpha_i;\ F),$$

with relations

$$(3)\qquad \alpha_i(m_i)-\alpha_k(m_k)=0(\forall\, i,k\in I).$$

The relations (1) are related with a system of linear equations

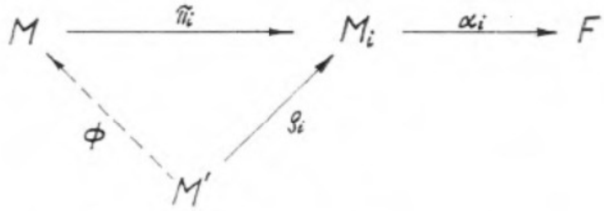
$$(4)\qquad \sum_{i\in I} r_{ji}x_i=0,\quad j\in J,$$

where $x_i=\alpha_i(m_i)$, $i\in I$. This system is a system of *homogeneous linear equations* over F, so we have always the zero solution of (4). If $K_i=\operatorname{Ker}(\alpha_i)$, $i\in I$, then M contains the direct product $K=\prod_{i\in I} K_i$ as a submodule. If therefore the number of $K_i\neq 0$ is infinite, M cannot be the direct sum of the M_i.

Therefore that M is the direct sum $\bigoplus_{i\in I} M_i$, it is necessary that $r_{ji}\alpha_i(m_i)=0$ for all i, j, and all m_i. With this condition only, M is still $\prod_{i\in I} M_i$, so I must be finite; according to our assumptions about the terms $r_{ji}\alpha_i(m_i)$ in (1), we can in this case omit the relations (1). Concerning the solutions of the equations (4): If $X=\langle ..., x_i, ...\rangle_{i\in I}, g_j=\sum_{i\in I} r_{ji}x_i$, and $Y=\langle ..., g_j, ...\rangle_{j\in J}$, then the solutions of (4) correspond in a one to one way with the elements of the R-module $\mathrm{Hom}_R(X/Y, F)$. Indeed, if $(..., f_i, ...)$ satisfies (4), then there is an R-homomorphism $\varphi: X/Y\to F$, determined by $\varphi(\bar{x}_i)=\varphi(x_i+Y)=f_i(i\in I)$. Conversely, if $\psi\in\mathrm{Hom}_R(X/Y, F)$, such that $\psi(\bar{x}_i)=\psi(x_i+Y)=f_i(i\in I)$, then $x_i=f_i(i\in I)$ is a solution of (4). If $\varphi\in\mathrm{Hom}_R(X/Y, F)$ determines the solution $x_i=f_i(i\in I)$ of (4), then $r\varphi$ determines the solution $rx_i=rf_i(i\in I)$.

A subproduct has the following *universal* property:

Theorem 1.1. *Let the subproduct* M *be defined by* (2) *and let* π_i *be the canonical projection* $\pi_i: M\to M_i(i\in I)$, *then* $\sum_{i\in I} r_{ji}\alpha_i\pi_i=0(\forall j\in J)$. *Conversely, if* M' *is an* R*-module,* $\varrho_i: M'\to M_i(\forall i\in I)$ *an* R*-homomorphism, with* $\sum_{i\in I} r_{ji}\alpha_i\varrho_i=0(\forall j\in J)$, *then there exists a unique* $\emptyset: M'\to M$, *satisfying* $\pi_i\Phi=\varrho_i(\forall i\in I)$.



Proof. If $\varrho_i(m')=m_i(i\in I)$, $\sum_{i\in I} r_{ji}\alpha_i(m_i)=0$, then the element $m=(m_i)_{i\in I}\in M$ and we define $\emptyset$ by $\emptyset(m')=m$; then $\pi_i\emptyset=\varrho_i$. If $\pi_i\emptyset=\pi_i\emptyset'$, then $\pi_i(\emptyset-\emptyset')m'==0(\forall i\in I)$, or $\emptyset(m')=\emptyset'(m')$. The solutions of the equations (4) form an R-module

$$S=\{(..., f_i, ...)|\sum_{i\in I} r_{ji}f_i=0;\ j\in J\};$$

with (f_i) and (f_i'), we know that (f_i+f_i') and (rf_i) are also solutions of (4). For *fixed* $i\in I$ the components f_i of the solutions form a submodule F_i of F. Therefore S is a subdirect product

$$S=\underset{i\in I}{\otimes} F_i,$$

uniquely determined by the subproduct (2).

If $N_i=\alpha_i^{-1}F_i(i\in I)$, then we see that the subproduct M is a subdirect product of the R-modules $N_i(i\in I)$:

$$M=\underset{i\in I}{\otimes} N_i\left(\alpha_i;\ F_i;\ \sum_{i\in I} r_{ji}\alpha_i(n_i)=0;\ j\in J\right). \tag{5}$$

If we have a set of subproducts $M^{(j)}$ with the *same* set $\{M_i; \alpha_i; F\}$, but evt. with different relation systems, then it is clear that the systems of relations *together* define a subproduct

$$M = \bigcap_j M^{(j)} \subset \prod_{i \in I} M_i.$$

This means that the subproduct, defined by (2) can be considered as the *intersection* of the *one-relation* subproducts

$$(6) \qquad M^{(j)} = \{M_i; \alpha_i; F; \sum_i r_{ji}\alpha_i(m_i) = 0, \text{ fixed } j \in J\}, \quad M = \bigcap_j M^{(j)}.$$

The solutions of (4) correspond in a one to one way with the elements of $\mathrm{Hom}_R(X/Y, F)$; now $M = \bigcap_j M^{(j)}$, where $M^{(j)}$ is defined by only one relation $\sum_i r_{ji}\alpha_i(m_i) = 0$ (j fixed).

The elements φ_j of $\mathrm{Hom}_R(X/Y_j, F)$, where $Y_j = \langle g_j \rangle$, correspond in a one to one way with the solutions $(\ldots, f_i^{(j)}, \ldots)$ of $g_j = 0$ by $\varphi_j(\bar{x}_i) = f_i^{(j)}$. It is clear that the set $(\ldots, \varphi_j, \ldots)_{j \in J}$ defines a solution of (4) (i.e. defines an element of $\mathrm{Hom}_R(X/Y, F)$) if for any two indices $j_1, j_2 \in J$ we have

$$\varphi_{j_1}(\ldots, \bar{x}_i, \ldots) = \varphi_{j_2}(\ldots, \bar{x}_i, \ldots).$$

If we denote by $H = \mathrm{Hom}_R(X/Y, F)$ and by $H^{(j)} = \mathrm{Hom}_R(X/Y_j, F), j \in J$, then H is a special subdirect product of the $H^{(j)}$ ($j \in J$), determined by the homomorphisms

$$\beta^{(j)} : \varphi_j \mapsto (\ldots, f_i^{(j)}, \ldots, f_k^{(j)}, \ldots).$$

§ 2. Special cases of subproducts

A. Let us consider the subproduct (2) of § 1 as the intersection of the one-relation subproducts $M^{(j)}$ (see § 1, (6)). Let

$$(7) \qquad r_{ji_1}\alpha_{i_1}(m_{i_1}) + \ldots + r_{ji_k}\alpha_{i_k}(m_{i_k}) = 0$$

be the corresponding relation of $M^{(j)}$, with corresponding equation

$$r_{ji_1}x_1 + \ldots + r_{ji_k}x_k = 0.$$

Then we have

Theorem 2.1. *$M^{(j)}$ is a subdirect product of the $M_i (i \in I)$, exactly if* $r_{ji_0}F \subset \sum_{i \neq i_0} r_{ji}F (\forall i_0; r_{j0} \neq 0)$. Indeed, this follows from the fact that for any $f \in F$ the element $r_{ji_0}f$ can be written as $\sum_{i \neq i_0} r_{ji}f_i$ for suitable $f_i \in F$.

Corollary 2.2. Let the subproduct M be given by (2) with coefficients $r_{ji} \neq 0$, then M can be considered as the intersection of R-modules $M^{(j)}$ ($j \in J$) — where each $M^{(j)}$ is a subdirect product of the M_i — if and only if $r_{ji_0}F \subset \sum_{i \neq i_0} r_{ji}F (\forall i_0 \in I, \forall j \in J)$.

B. In the special case that we have two R-modules $M_1, M_2, \alpha_i : M_i \to F (i = 1, 2)$, the two R-epimorphisms and the *only* relation $r_1\alpha_1(m_1) + r_2\alpha_2(m_2) = 0$, together with the conditions that $r_1F = r_2F$, then M is a subdirect product of M_1 and M_2.

But this subdirect product must be a special subdirect product of M_1 and M_2. If we define

$$N_1 = \{m_1 \in M_1 | (m_1, 0) \in M\}, \quad N_2 = \{m_2 \in M_2 | (0, m_2) \in M\},$$

then $N_1 = \{m_1 \in M_1 | r_1 \alpha_1(m_1) = 0\}$, etc.

This implies that there is a module $F_1 \subset F$ consisting of those $f_1 \in F$ with $r_1 f_1 = 0$. In the same way there exists a submodule $F_2 \subset F$ with $F_2 = \{f_2 \in F | r_2 f_2 = 0\}$ and

$$\alpha_1 N_1 = F_1, \quad \alpha_2 N_2 = F_2.$$

We prove that

$$M_1/N_1 \cong M_2/N_2.$$

Indeed, define a map $\varphi: M_1/N_1 \to M_2/N_2$ in the following way: if $m = (m_1, m_2) \in M$, then define

$$\varphi : m_1 + N_1 \mapsto m_2 + N_2;$$

if $m' = (m_1, m_2') \in M$, then $m_2 - m_2' \in N_2$ and $m_2 + N_2 = m_2' + N_2$.

Since M is a subdirect product of M_1 and M_2, every m_2 occurs, so φ is „onto". Ker $(\varphi) = \{(m_1, m_2) \in M | m_2 \in N_2\}$. Then it follows that $m_1 \in N_1$, hence Ker (φ) is the zero element of M_1/N_1; this proves that $M_1/N_1 \cong M_2/N_2$. Since

$$M_1/N_1 \cong M_1/K_1/N_1/K_1 \cong F/F_1, \quad \text{and} \quad M_2/N_2 \cong F/F_2,$$

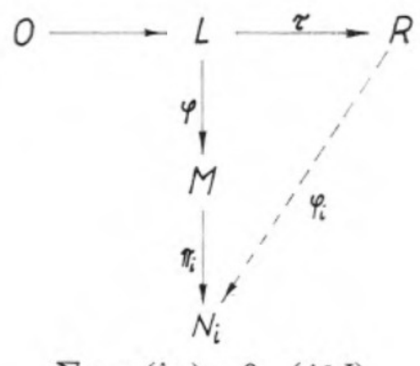
we have

$$F/F_1 \cong F/F_2.$$

Moreover M consists of all pairs (m_1, m_2) for which the natural homomorphisms $M_1 \to M_1/N_1$, $M_2 \to M_2/N_2$ map m_1 and m_2 upon corresponding cosets.

C. **Theorem 2.3.** *Suppose that we have represented the subproduct* M(of § 1,(2)) *in the form* (5) *as a subdirect product of the* N_i; *then the following statement holds: if the* $N_i (i \in I)$ *are injective and* F *is torsionfree, then* M *is an injective* R*-module.*

Proof. Let $L (\neq 0)$ be an ideal of the ring R, and $\varphi: L \to M$, $\pi_i: M \to N_i (i \in I)$ R-homomorphisms, where π_i is the canonical projection, then there exists a $\varphi_i: R \to N_i (i \in I)$, such that $\varphi_i \tau = \pi_i \varphi$. Suppose $\varphi_i(1) = n_i$, then for $\lambda \in L$, we have $\varphi_i(\lambda) = \lambda n_i$. If $\varphi(\lambda) = m \in M$, then $\varphi(\lambda) = (\ldots, \lambda n_i, \ldots, \lambda n_k, \ldots)$, and this element belongs to $M = \bigtimes_{i \in I} N_i$, such that the images $\alpha(\lambda n_i)$ satisfy the relation



$$\sum_{i \in I} r_{ji} \alpha_i(\lambda n_i) = 0 \quad (j \in J),$$

hence

$$\lambda\Big(\sum_{i\in I} r_{ji}\alpha_i(n_i)\Big) = 0 \quad (\forall \lambda\in L;\ j\in J).$$

Since F is torsionfree, $L\neq 0$, we find

$$\sum_{i\in I} r_{ji}\alpha_i(n_i) = 0 \quad (j\in J),$$

and that implies that the element $m_0=(\dots, n_i, \dots)\in M$, and moreover, that $\varphi(\lambda)=\lambda m_0$ proving that M is injective.

Remark 1. If — in particular — $N_i=M_i$, and therefore $F_i=F(\forall i\in I)$, then the injectivity of the M_i and the torsionfreeness of F implies the injectivity of M.

Remark 2. In the same way one proves:

Corollary. If M_i is injective $(\forall i\in I)$ and F is torsionfree, then M is injective.

D. *The case of a subdirect product.*

In general, the subproduct M (see § 1, (2)) is not a subdirect product of the $M_i(i\in I)$. From § 1 we know that the solutions of the corresponding equations (4) over the module F form an R-module $S=\big\{(\dots,f_i,\dots)\mid \sum_{i\in I} r_{ji}f_i=0;\ j\in J\big\}$, and moreover $S=\widehat{\times}_{i\in I} F_i$. A necessary and sufficient condition therefore that M is a subdirect product of the M_i is, that $F_i=F(\forall i\in I)$. One can express this (necessary and sufficient) condition in another way:

(i) M is a subdirect product of the M_i, if and only if — for every $k\in I$ and for every $f\in F$ — the equations

$$\text{(9)} \qquad \sum_{i\neq k} r_{ji}x_i = -r_{jk}f \quad (j\in J)$$

are solvable in F. Equivalent with this condition (i) is the condition

(ii) For every $k\in I$ and for every $f\in F$, there exists an element $\varphi\in \mathrm{Hom}_R(X/Y, F)$ such that

$$\varphi(\bar{x}_k) = \varphi(x_k+Y) = f.$$

From the theory of linear equations over an R-module we know that a necessary condition for the solvability of the system (9) is the compatibility of the system (9) *for every* $k\in I$ and *every* $f\in F$.

Theorem 2.4. *If the systems (9) are compatible for every $k\in I$ and every $f\in F$, and F is injective, then M is a subdirect product of the M_i.*

§ 3. Other subproducts

1. Wich subproducts (2) describe the cartesian product $M=\prod_{i\in I} M_i$? In that case *all* elements $m=(m_i)_{i\in I}\in \prod_{i\in I} M_i$ have to satisfy the relations (1). Taking for $m=(\dots, 0, m_i, 0, \dots)$ we find as necessary conditions

$$\text{(10)} \qquad r_{ji}\alpha_i(m_i) = 0$$

and we have to take care of the fact that $\alpha_i(m_i)$ runs through all elements of F, i.e. $r_{ji}f=0(\forall i\in I, j\in J)$. The relations (10) can be satisfied if either

(11) (i) $F = 0$;

(11ᵃ) (ii) $r_{ji}\in \mathrm{Ann}_R(F)$ (for all $i\in I, j\in J$).

The conditions (11), resp. (11ᵃ) guarantee also that M is the cartesian product.

2. If M and M' are two subproducts with the same $\{M_i, \alpha_i, F\}_{i\in I}$, can they describe the same subproduct $M=M'$? The corresponding systems of equations are

$$\text{(i)} \sum_{i\in I} r_{ji}x_i = 0 \quad (j\in J), \quad \text{resp.} \quad \sum_{i\in I} r'_{j'i}x_i = 0 \quad (j'\in J') \text{ (ii).}$$

If $m=(m_i)_{i\in I}\in M$, then the $\{\alpha_i(m_i)\}_{i\in I}$ have to satisfy (i) and (ii) and conversely. That means that the systems (i) and (ii) must have the same solutions $x=(x_i)_{i\in I}$. Denoting by $Y=\langle ..., g_j, ...\rangle$, $g_j=\sum_{i\in I} r_{ji}x_i$, $Y'=\langle ..., g'_j, ...\rangle$, $g'_j=\sum_{i\in I} r'_{j'i}x_i$, we have: if $Y=Y'$, then the solutions of the corresponding systems $\sum_i r_{ji}x_i=0(j\in J)$, resp. $\sum_i r'_{j'i}x_i=0$ $(j'\in J')$ coincide and that implies $M=M'$. Conversely: if M and M' coincide, then the corresponding equations of any of the two systems are consequences of the equations of the other, i.e. $Y=Y'$. This means: $Y=Y'$ is the necessary and sufficient condition therefore that $M=M'$.

§ 4. The case of the special subproduct

Suppose that $M=\{M_i; \alpha_i; F; \sum_{i\in I} r_{ji}\alpha_i(m_i)=0\}_{j\in J}$ is a given subproduct; we define

$$C_i = M\cap M_i = \{m\in M|\pi_k m = 0,\ k\neq i\};$$

C_i is the submodule of M with elements $m=(..., 0, m_i, 0, ...)$. Using the given relations, we find for the components m_i of $m\in C_i$:

$$r_{ji}\alpha_i(m_i) = 0(\forall j\in J). \tag{12}$$

These relations can be satisfied in the following way:

(i) F=0; in that case $M=\prod_{i\in I} M_i$, $C_i=M_i(\forall i\in I)$;

(ii) we denote — for every fixed $i\in I$ — by S_i the subset of R defined by $S_i=\{..., r_{ji}, ...\}_{j\in J}$; then there is a submodule $V_i\subset F$, defined by

$$V_i = (0:_F S_i) = \{f\in F|S_i f = 0\}. \tag{13}$$

V_i consists of the elements $v=\alpha_i(m_i)$, satisfying the condition (12), and that means that there is a submodule $C_i=M\cap M_i$, with $\alpha_i C_i=V_i(i\in I)$; we have

$$K_i = \mathrm{Ker}\,\alpha_i\subset C_i\subset M_i.$$

Let C be the submodule of M, defined by

$$C = \prod_{i\in I} C_i; \tag{14}$$

we remind that each equation $\sum_{i\in I} r_{ji}\alpha_i(m_i)=0$ is a finite sum of expressions $r_{ji}\alpha_i(m_i)$ and for every $m_i\in C_i$, we have $r_{ji}\alpha_i(m_i)=0$. Furthermore we define the submodules

(15) $$D_i\subset M \quad (i\in I),$$

consisting of the elements $d=(d_i)_{i\in I}\in M$ with $d_i=0$. Let $N_i=\pi_i M\subset M_i$, then

$$M/D_i \cong N_i.$$

We now suppose that M *satisfies the relations*

(16) $$D_i\subset C(\forall\, i\in I).$$

This implies

$$M/C \cong M/D_i/C/D_i \cong N_i/C_i;$$

using the property that $F_i=\alpha_i(N_i)$, we have

(17) $$N_i/C_i \cong N_i/K_i/C_i/K_i \cong F_i/V_i \quad (i\in I).$$

This means that every M_i contains a submodule N_i such that the quotients N_i/C are „invariant"; i.e. if we denote M/C by $\tilde F$, then we have, for all $i\in I$,

$$N_i/C_i \cong \tilde F.$$

If $m=(\dots, m_i, \dots)\in M$, $\pi_i(m)=m_i\in N_i$, then we define $\beta_i:N_i\to\tilde F$ by

(18) $$\beta_i(m_i) = m+C, \quad \text{if} \quad \pi_i(m) = m_i.$$

Then β does not depend upon the choice of $m\in M$; for if $\pi_i(m)=\pi_i(m')$, then $m-m'\in D_i\subset C$, hence $m+C=m'+C$. If therefore $m=(m_i)_{i\in I}\in M$, it follows from (18), that

(19) $$\dots = \beta_i(m_i) = \dots = \beta_j(m_j) = \dots = m+C;$$

this means that there is an R-module $\tilde F$ and epimorphisms $\beta_i:N_i\to\tilde F(i\in I)$, such that *all* elements $m\in M$ satisfy the property of a special subdirect product. If, conversely, $n=(n_i)_{i\in I}\in\prod_{i\in I} N_i$ has the property, that $\dots=\beta_i n_i=\dots=\beta_j n_j=\dots$, then $n\in M$. Indeed, $\beta_i n_i=n'+C$ for some $n'\in M$, only if $\pi_i n'=n_i$; but then $n=n'\in M$. Using the notations we have defined here we have proved the

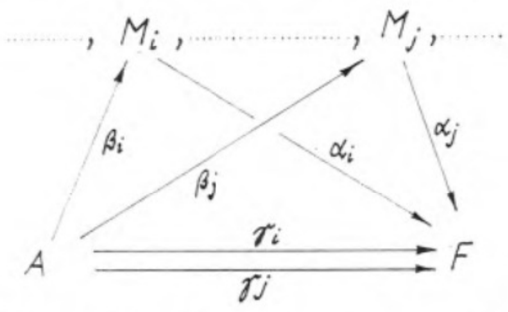
Theorem 4.1. *Under the conditions* $D_i\subset C(i\in I)$, *the subproduct* $M=\{M_i;\alpha_i;F;\sum_{i\in I} r_{ji}\alpha_i(m_i)=0\}$ *is a special subdirect product* $M=\widehat{\times}_{i\in I} N_i(\beta_i;\tilde F)$.

If for at least one $i\in I$ we have $C_i=M\cap M_i=M_i$, then $N_i=M_i$ for all $i\in I$, and $F=M/C=0$; in that case $M=\prod_{i\in I} M_i$. If however for at least one $i\in I$ we have $C_i\subseteqq N_i$, then this relation holds for all $i\in I$.

Conversely, suppose that the subproduct M (see (2)) has the property that M is a special subdirect product of the submodules $N_i\subset M_i$ (where $N_i=\pi_i M$) with respect to epimorphisms $\beta_i:N_i\to\tilde F$, then $M=\{m=(m_i)_{i\in I}, m_i\in N_i|\beta m_i=\beta_k m_k(i,k\in I)\}$. For the elements $d\in D_i$, $d=(d_i)_{i\in I}\in M$, we have then $\beta_i n_i=\beta_k n_k=0$ for all $k\in I$. Since $C=\{c=(c_i)_{i\in I}|\beta_i c_i=0, \forall\, i\in I\}$, we have $D_i\subset C(\forall\, i\in I)$.

§ 5. The strukture of $\mathrm{Hom}_R(A, M)$, if M is a subproduct

Let $M=\{M_i; \alpha_i; F; \sum_{i\in I} r_{ji}\alpha_i(m_i)=0; j\in J\}$ be any subproduct, then we want to study the structure of $\mathrm{Hom}_R(A, M)$ for any R-module A.



If $\beta\in\mathrm{Hom}_R(A, M), a\in A$, then $\beta(a)=(\dots, m_i, \dots, m_j, \dots)\in M$, and $\sum_{i\in I} r_{ji}\alpha_i(m_i)=0$; β induces a system of R-homomorphisms

$$\beta_i : A \to M_i, \quad \beta_i = \pi_i\beta \quad (\forall\, i\in I),$$

where π_i is the canonical projection $\pi_i : M\to M_i (i\in I)$. Hence $\beta_i(a)=m_i (i\in I)$. Substitution in $\sum_{i\in I} r_{ji}\alpha_i(m_i)=0$ gives

$$\sum_{i\in I} r_{ji}\alpha_i\beta_i(a) = 0 \quad (j\in J,\ \forall\, a\in A). \tag{2}$$

Defining $\alpha_i\beta_i=\gamma_i$, we see that

$$\sum_{i\in I} r_{ji}\gamma_i(a) = 0 \quad (j\in J,\ \forall\, a\in A). \tag{3}$$

This implies that if we have a subproduct M and $\beta\in\mathrm{Hom}_R(A, M)$, then β induces a set of R-homiomorphisms $\beta_i : A\to M_i$, such that (2) holds. Moreover β induces a system $(\dots, \gamma_i, \dots)$, $\gamma_i\in\mathrm{Hom}_R(A, F), \gamma_i=\alpha_i\beta_i, i\in I$, such that the relations

$$\sum_{i\in I} r_{ji}\gamma_i(a) = 0 \quad (j\in J) \tag{4}$$

hold for all $a\in A$. Conversely: suppose we have a set of R-homomorphisms $\beta_i\in\in\mathrm{Hom}_R(A, M_i)$, $i\in I$, such that for all $a\in A$ we have the identities $\sum_{i\in I} r_{ji}\alpha_i\beta_i(a)=0$; then the system $(\dots, \beta_i, \dots)$ determines in a unique way a homomorphism $\beta : A\to M$ by $\beta(a)=(\dots, \beta_i(a), \dots)$ and this element belongs to M, since (2) is satisfied. Moreover we learn from (4), that — for a certain $j\in J$ — we have

$$r_{j1}\gamma_1(a)+r_{j2}\gamma_2(a)+\dots+r_{jk_j}\ \ \gamma_{k_j}(a) = 0 \quad (\forall\, a\in A),$$

i.e.

$$r_{j1}\gamma_1+r_{j2}\gamma_2+\dots+r_{jk_j}\gamma_{k_j} = 0,$$

(using the fact that R is commuative!); that means — for all $j \in J$ — we have the identities

(5) $$\sum_{i \in I} r_{ji} \gamma_i = 0 \quad (j \in J).$$

The result can be expressed as follows:

Theorem 5.1. *If M is a subproduct, defined by* (1), *and A is any R-module, then $\mathrm{Hom}_R(A, M)$ is a subproduct of the $\mathrm{Hom}_R(A, M_i)$, determined by*

$$\mathrm{Hom}_R(A, M) \cong \{\mathrm{Hom}_R(A, M_i);\ (\alpha_i)_*;\ \mathrm{Hom}_R(A, F);\ \sum_{i \in I} r_{ji}(\alpha_i)_* = 0;\ j \in J\},$$

where $(\alpha_i)_* = \alpha_i \beta_i$.

The result is an exchange of the operation Hom with the operation of taking subproducts; in fact, it means that any $\beta \in \mathrm{Hom}_R(A, M)$ is determined by a set $(\beta_i)_{i \in I}$, $\beta_i \in \mathrm{Hom}_R(A, M_i)$, such that the relations $\sum_{i \in I} r_{ji} \alpha_i \beta_i = 0 (j \in J)$ are satisfied.

Remark. We know that the structure of a subdirect product of n modules $M_i (n > 2)$ is rather complicated. Prof. L. FUCHS sent me the following remark, proving that a subdirect product of finitely many modules can be described by a subproduct.

Let M be a subdirect product $M = \underset{i=1}{\overset{n}{\tilde{\times}}} M_i$ of the finitely many modules $M_1, \ldots, M_n$. Then define $F = \bigoplus_{i=1}^{n} M_i / M$ and let $\alpha_i ; M_i \to F$ be the maps defined by $\alpha_i : m_i \mapsto m_i + M$. Then M can be recaptured by using the single equation $\sum_{i=1}^{n} \alpha_i m_i = 0$.

I am very grateful to Professor L. Fuchs for his kind help in reading the manuscript and for his useful remarks.

DEPARTMENT OF MATHEMATICS OF THE TECHNICAL
UNIVERSITY, DELFT.

(Received May 29, 1975.)