On polynomial regression of polynomial and linear statistics

By BÉLA GLEVITZKY (Debrecen)

1. Introduction

In [3], YU. V. LINNIK and A. A. ZINGER proved the following theorem:

Let $\xi_1, \ldots, \xi_n$ be a sample from random variable ξ and assume that ξ has moments up to order m. Let the adjoint polynomial of the polynomial statistic

$$P = \sum A_{\nu_1 \ldots \nu n} \xi_1^{\nu_1} \ldots \xi_n^{\nu_n}$$

be a regular polynomial statistic of degree p_1 and order $m | m \leqq p_1 |$. Here and in the sequel the summation is extended over all nonnegative integers $\nu_1, \ldots, \nu_n$ which satisfy the relation

$$\nu_1 + \ldots + \nu_n \leqq p_1. \tag{1.1}$$

If P has constant regression on $\Lambda = \xi_1 + \ldots + \xi_n$ and $m \geqq n-1$ then the characteristic function $f(t)$ of ξ is an entire function.

In [2], B. GYIRES generalized this theorem in two direction. First he supposed only that $\xi_1, \ldots, \xi_n$ are independent random variables instead of being a sample, consisting of independently and identically distributed random variables. Secondly he avoided the condition $m \geqq n-1$. It is well known that this condition strongly narrows down the applicability of the theorem of Linnik and Zinger.

In this paper we also avoid the condition $m \geqq n-1$. We suppose that $\xi_1, \ldots, \xi_n$ are independent but not necessarily identically distributed random variables and that they have moments of all orders. We give an extension the theorem of Linnik and Zinger in the case, when P has regression of order $r\,(0 \leqq r \leqq m-1)$ on $\Lambda = \sum_{j=1}^{n} \alpha_j \xi_j$ $\alpha_j \neq 0$, $\alpha_j \in R_1, j = 1, 2, \ldots, n$. We shall use the following notations, definitions and results.

Let R_n be the n-dimensional space with row vectors as elements. If $v \in R_n$ then v^* stands for the transpose of the vector v.

$a \in R_n$ is the vector with all components equal to 1.

Let the components of the random vector-variable $\xi=(\xi_j)\in R_n$ be independent random variables. Denote the distribution function of ξ_j by $F_j(x)$ $(j=1, 2, ..., n)$ and assume that $F_j(x)$ has moments up to order m. Let

$$P = \sum A_{\nu_1 \dots \nu_n} \xi_1^{\nu_1} \dots \xi_n^{\nu_n}$$

be a polynomial statistic of degree p_1.

Definition 1.1. The statistic P is said to be a common polynomial statistic of degree p_1 and order $m|m\leqq p_1|$ if the following conditions are satisfied:

a) The statistic P is a nonnegative polynomial;

b) No exponent in P exceeds m;

c) P contains the m-th power of each variable. (If the random variables $\xi_1, ..., \xi_n$ are identically distributed then it is sufficient to assume that the polynomial P of degree p_1 and order $m\leqq p_1$ contains the m-th power of at least one variable. In this case P is a regular polynomial statistic of degree p_1 and order m.)

Definition 1.2. A characteristic function $f(t)$ is said to be an analytic characteristic function if there exists a function $A(z)$ of the complex variable z which is regular in the circle $|Z|<\varrho$ $(\varrho>0)$ and a constant $\Delta>0$ such that $A(t)=f(t)$ for $|t|<\Delta$.

Lemma 1.1. ([1], p. 89, lemma 5.3.4.) *If a characteristic function $f(z)$ is regular in a neighbourhood of the origin then it is also regular in a horizontal strip and it can be represented in this strip by the Fourier integral $f(z)=\int_{-\infty}^{\infty} e^{izx}\, dF(x)$ where $F(x)$ is the distribution function corresponding to $f(z)$. This strip is either the whole plane, or it has one, or two, horizontal boundary lines.*

Definition 1.3. If the strip is the whole plane then $f(z)$ is called an entire characteristic function.

Definition 1.4. Consider two random variables ξ and η and assume that the conditional expectation $E(\eta|\xi)$ exists. We say that η has polynomial regression of order r on ξ if the relation

$$E(\eta|\xi) = \beta_0+\beta_1\xi+\ldots+\beta_r\xi^r$$

holds almost everywhere.

Assume that the first moments of η and the r-th moment of ξ exist. Then

$$E(\eta) = \beta_0+\beta_1 E(\xi)+\ldots+\beta_r E(\xi^r).$$

Theorem 1.1. ([1], p. 103, Theorem 6.1.1.) *Let ξ and η be two random variables and assume that the expectations $E(\eta)$ and $E(\xi^r)$ exist where r is a nonnegative integer. The random variable η has polynomial regression of order r on ξ if and only if the relation*

$$E(\eta e^{it\xi}) = \sum_{j=0}^{r} \beta_j E(\xi^j e^{it\xi})$$

holds for all real t. Here the β_j are real constants.

2. A new version of Linnik—Zinger's theorem

First we prove the following theorem.

Theorem 2.1. *Let the components of the random vector-variable* $\zeta=(\xi_j)\in R_n$ *be independent but not necessarily identically distributed random variables. Let the distribution function of* ξ_j *be denoted by* $F_j(x)$ *and assume that* ξ_j *has moments of all orders,* $j=1, 2, \ldots, n$. *Let* $\Lambda=\zeta a^*$ *and let*

$$P = \sum A_{v_1 \ldots v_n} \xi_1^{v_1} \ldots \xi_n^{v_n}$$

be a common polynomial statistic of degree p_1 *and order* m $(m \leqq p_1)$. *If* P *has polynomial regression of order* $r\,|0 \leqq r \leqq m-1|$ *on* Λ *then the characteristic function* $f_j(t)$ *of* $F_j(x)$ *is an entire characteristic function.*

To prove theorem 2.1. the following lemma is needed.

Lemma 2.1. ([2], lemma 3.1.) *Let* α *be a positive constant. Let* s *and* k *be positive integers,* $s \geqq k$. *Put*

$$S_{k\alpha}(s) = \sum \left(\frac{s}{j_1 \ldots j_k}\right)^{1+\alpha}$$

where the summation is extended over all positive integers $j_1, \ldots, j_k$ *which satisfy the relation* $j_1+\ldots+j_k=s$. *Then*

$$S_{k\alpha}(s) = (2^{2+a} \cdot s_\alpha)^{k-1}$$

where s_α *is the sum of the series*

$$1+\left(\frac{1}{2}\right)^{1+\alpha}+\left(\frac{1}{3}\right)^{1+\alpha}+\ldots$$

PROOF *of theorem 2.1.*

Since P has polynomial regression of order $r\,|0 \leqq r \leqq m-1|$ on Λ it is easily seen that the characteristic functions $f_j(t)$ of ξ_j $|j=1, 2, \ldots, n|$ satisfy the following equation:

$$(2.1) \qquad \int_{R_n} P(x) e^{itxa^*} dF_1(x_1) \ldots dF_n(x) = \sum_{p=0}^{r} \gamma_p \int_{R_n} (xa^*)^p e^{itxa^*} dF_1(x_1) \ldots dF_n(x_n).$$

Since ξ_j $|j=1, 2, \ldots, n|$ has moments of all orders, we may differentiate (2.1) any number of times. By differentiating both sides of (2.1) N-times then putting $t=0$ we obtain

$$(2.2) \qquad \int_{R_n} P(x)(xa^*)^N dF_1(x_1) \ldots dF_n(x_n) = \sum_{p=0}^{r} \gamma_p \int_{R_n} (xa^*)^{N+p} dF_1(x_1) \ldots dF_n(x_n).$$

Denote the k-th absolut moment of $F_j(x)$ by

$$(2.3) \qquad \beta_k^{(j)} = \int_{-\infty}^{\infty} |x|^k dF_j(x_j) \quad |j = 1, \ldots, n|.$$

First we show that the inequality

$$\beta_k^{(j)} \leqq \frac{k!}{k^{1+\alpha}} M^k \tag{2.4}$$

holds for any positive integer k where α and M are positive constants which will be chosen later. We prove (2.4) by induction.

Since P is a common polynomial statistic it is nonnegative and so its order m is necessarily an even integer. Therefore we can write $|m \geqq 2|$

$$P = P_0(\xi_2, \ldots, \xi_n)\xi_1^m + \ldots + P_m(\xi_2, \ldots, \xi_n) \tag{2.5}$$

where $P_0(\xi_2, \ldots, \xi_n)$ is also a nonnegative polynomial in the $n-1$ variables $\xi_2, \ldots, \xi_n$. First let N (and thus k) be an even integer. We use the inequalities

$$\big||x_1| - |x_2 + \ldots + x_n|\big| \leqq |xa^*| \leqq |x_1| + \ldots + |x_n| \tag{2.6}$$

and obtain from (2.2) that

$$\begin{aligned} &\int_{R_n} P(x)(|x_1| - |x_2 + \ldots + x_n|)^N \, dF_1(x_1) \ldots dF_n(x_n) \leqq \\ &\leqq \sum_{p=0}^{r} |\gamma_p| \int_{R_n} \left(\sum_{j=1}^{n} |x_j| \right)^{N+p} dF_1(x_1) \ldots dF_n(x_n) \end{aligned} \tag{2.7}$$

Theorem 2.1. is trivial if the ξ_j's are bounded random variables. So assume, from now on, that at least one of the random variables $\xi_1, \ldots, \xi_n$ is not bounded. Let

$$1 - F_1(x-0) + F_1(-x) > 0. \tag{2.8}$$

It is always possible to find a bounded region $\Omega \subset R_{n-1}$ such that

$$\int_{\Omega} dF_2(x_2) \ldots dF_n(x_n) = c_1 > 0 \tag{2.9}$$

$$\min_{\Omega} P_0(x_2, \ldots, x_n) = c_2 > 0 \tag{2.10}$$

We can find positive constants C_3, C_4, C_5 such that the relations

$$P(x) \geqq c_4 |x_1|^m \tag{2.11}$$

$$|xa^*| \geqq c_5 |x_1| \tag{2.12}$$

are satisfied, provided that $|x_1| > c_3$ and $(x_2, \ldots, x_n) \in \Omega$. The integrand on the left hand side of (2.7) is nonnegative, so that

$$\begin{aligned} &\int_{|x_1| > c_3} \int_{\Omega} x_1^m (|x_1| - |x_2 + \ldots + x_n|)^N \, dF_1(x_1) \ldots dF_n(x_n) \leqq \\ &\leqq \sum_{p=0}^{r} q_p \int_{R_n} \left(\sum_{j=1}^{n} |x_j| \right)^{N+p} dF_1(x_1) \ldots dF_n(x_n). \end{aligned} \tag{2.13}$$

Let

$$b_j = \int_{\Omega} |x_2 + \ldots + x_n|^j \, dF_2(x_2) \ldots dF_n(x_n), \tag{2.14}$$

$$I_0 = \int_{|x_1| \leqq c_3} \int_{\Omega} x_1^m (|x_1| - |x_2 + \ldots + x_n|)^N \, dF_1(x_1) \ldots dF_n(x_n), \tag{2.15}$$

$$\Omega_0 = \{x \mid |x_1| \leqq c_3, \ (x_2, \ldots, x_n) \in \Omega, \ x \in R_n\}, \tag{2.16}$$

$$c_8 = \max_{\Omega_0} \{|x_1|, \ \big||x_1| - |x_2 + \ldots + x_n|\big|\}. \tag{2.17}$$

Now we see that

$$I_0 \leqq b_0 C_8^{m+N}. \tag{2.18}$$

It follows from (2.13)—(2.18) that

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{\Omega} x_1^m (|x_1| - |x_2 + \ldots + x_n|)^N \, dF_1(x_1) \ldots dF_n(x_n) = \\ &= \sum_{j=0}^{N} \binom{N}{j} (-1)^j \int_{-\infty}^{\infty} |x_1|^{N+m-j} \, dF_1(x_1) \int_{\Omega} |x_2 + \ldots + x_n|^j \, dF_2(x_2) \ldots dF_n(x_n) = \\ &= \sum_{j=0}^{N} \binom{N}{j} \beta_{N+m-j}^{(1)} b_j (-1)^j \leqq b_0 C_8^{m+N} + \\ &+ \sum_{p=0}^{r} q_p \sum_{j_1^{(p)} + \ldots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \ldots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \ldots \beta_{j_n^{(p)}}^{(n)}. \end{aligned} \tag{2.19}$$

Therefore

$$\begin{aligned} \beta_{N+m}^{(1)} b_0 &\leqq \sum_{j=1}^{N} \binom{N}{j} \beta_{N+m-j}^{(1)} b_j + b_0 C_8^{m+N} + \\ &+ \sum_{p=0}^{r} q_p \sum_{j_1^{(p)} + \ldots + j_n^{(p)} = N+p} \frac{(N+p)!}{j_1^{(p)}! \ldots j_n^{(p)}!} \beta_{j_1^{(p)}}^{(1)} \ldots \beta_{j_n^{(p)}}^{(n)}. \end{aligned} \tag{2.20}$$

It follows from (2.14) that there exists a positive constant b such that

$$b_j \leqq b_0 b^j. \tag{2.21}$$

Thus there exists a positive constant M_0 such that

$$\beta_k^{(j)} \leqq \frac{k!}{k^{1+\alpha}} M_0^k \tag{2.22}$$

where $k=1, 2, \ldots, m-1; j=1, \ldots, n$ and $\alpha>0$. Assume now that (2.4) holds for $k=1, 2, \ldots, N+m-1$ and for some $M \geqq M_0$ which will be selected later. Then we have to show that (2.4) is valid for $k=N+m$. Using the inequality (2.4) for $k=1, 2, \ldots, N+m-1$, we obtain from (2.20) and (2.21) that

$$(2.23)\quad \begin{aligned} \beta^{(1)}_{N+m} b_0 \leqq \sum_{j=1}^{N} \binom{N}{j} \frac{(N+m-j)!}{(N+m-j)^{1+\alpha}} M^{N+m-j} b_0 b^j + \\ + b_0 C_8^{m+N} + \sum_{p=0}^{r} q_p \sum_{j_1^{(p)}+\ldots+j_n^{(p)}=N+p} \frac{(N+p)!}{j_1^{(p)}! \ldots j_n^{(p)}!} \beta^{(1)}_{j_1^{(p)}} \ldots \beta^{(n)}_{j_n^{(p)}}. \end{aligned}$$

Note first that

$$(2.24)\quad \binom{N}{j} (m+N-j)! < \frac{(m+N)!}{j!}$$

so that

$$(2.25)\quad \begin{aligned} \beta^{(1)}_{N+m} \leqq \sum_{j=1}^{N} \frac{(N+m)!}{j!\,(N+m-j)^{1+\alpha}} M^{N+m-j} b^j + \\ + C_8^{m+N} + \sum_{p=0}^{r} q'_p \sum_{j_1^{(p)}+\ldots+j_n^{(p)}=N+p} \frac{(N+p)!}{j_1^{(p)}! \ldots j_n^{(p)}!} \beta^{(1)}_{j_1^{(p)}} \ldots \beta^{(n)}_{j_n^{(p)}} \end{aligned}$$

where $q'_p = q_p | b_0 \ |p=0, 1, \ldots, r|$.

Let $0<\alpha<\dfrac{3 \lg 2}{\lg 3}-1$ then $j^{1+\alpha} \leqq 2^j$ and if we apply lemma 2.1. and Schwartz's inequality, we obtain

$$\sum_{j=1}^{N} \frac{(N+m)!}{j!\,(N+m-j)^{1+\alpha}} M^{N+m-j} b^j \leqq$$

$$(2.26)\quad \begin{aligned} &\leqq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sum_{j=1}^{N} \left(\frac{b}{M}\right)^j \frac{1}{j!} j^{1+\alpha} \left(\frac{N+m}{j(N+m-j)}\right)^{1+\alpha} \leqq \\ &\leqq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sum_{j=1}^{N} \left(\frac{2b}{M}\right)^j \frac{1}{j!} \left(\frac{N+m}{j(N+m-j)}\right)^{1+\alpha} \leqq \end{aligned}$$

$$\leqq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{\sum_{j=1}^{N} \left[\frac{1}{j!}\left(\frac{2b}{M}\right)^j\right]^2} \cdot$$

$$\cdot \sqrt{\sum_{j=1}^{N} \left(\frac{N+m}{j(N+m-j)}\right)^{2(1+\alpha)}} \leqq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{e^{(2b/M)^2}-1} \cdot S_2(N+m) \leqq$$

$$\leqq \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \sqrt{e^{(2b/M)^2}-1} \cdot 2^{2+\alpha} \cdot s_\alpha,$$

and

$$\sum_{j_1^{(p)}+\ldots+j_n^{(p)}=N+p} \frac{(N+p)!}{j_1^{(p)}!\ldots j_n^{(p)}!}\beta^{(1)}_{j_1^{(p)}}\ldots\beta^{(n)}_{j_n^{(p)}} =$$

$$= \sum_{k=1}^{n} \sum_{1\le l_1^{(p)}<\ldots<l_k^{(p)}\le n} \sum_{j_1^{(p)}+\ldots+j_n^{(p)}=N+p} \frac{(N+p)!\,\beta^{(l_1)}_{j_1^{(p)}}\ldots\beta^{(l_k)}_{j_k^{(p)}}}{j_1^{(p)}!\ldots j_k^{(p)}!} \le$$

$$\le \frac{(N+p)!\,M^{N+p}}{(N+p)^{1+\alpha}} \sum_{k=1}^{n}\binom{n}{k} \sum_{j_1^{(p)}+\ldots+j_n^{(p)}=N+p} \frac{(N+p)^{1+\alpha}}{(j_1^{(p)})^{1+\alpha}\ldots(j_n^{(p)})^{1+\alpha}} =$$

$$= \frac{(N+p)!\,M^{N+p}}{(N+p)^{1+\alpha}} \sum_{k=1}^{n}\binom{n}{k} S_k(N+p) \le$$

$$\le \frac{(N+p)!}{(N+p)^{1+\alpha}} M^{N+p} \sum_{k=1}^{n}\binom{n}{k} (2^{2+\alpha}\cdot s_\alpha)^{k-1} = \tag{2.27}$$

$$= \frac{(N+p)!}{(N+p)^{1+\alpha}} M^{N+p} \frac{1}{2^{2+\alpha}\cdot s_\alpha}[(1+2^{2+\alpha}\cdot s_\alpha)^n-1] =$$

$$= \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{N+m}{N+p}\right)^{1+\alpha} \left(\frac{1}{M}\right)^{m-p} \frac{1}{(N+p+1)\ldots(N+m)}\cdot$$

$$\cdot \frac{1}{2^{2+\alpha}\cdot s_\alpha}[(1+2^{2+\alpha}\cdot s_\alpha)^n-1] <$$

$$< \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{1}{M}\right)^{m-p} \left(1+\frac{m-p}{N+p}\right)^{1+\alpha} \frac{1}{2^{2+\alpha}\cdot s_\alpha}[(1+2^{2+\alpha}\cdot s_\alpha)^n-1] \le$$

$$\le \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \left(\frac{1}{M}\right)^{m-p} (1+m)^{1+\alpha}.$$

Therefore from (2.25), (2.26) and (2.27) it follows that

$$\beta^{(1)}_{N+m} \le \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m} \Bigg[\sqrt{e^{(2b/M)^2}-1}\cdot 2^{2+\alpha}\cdot s_\alpha + \left(\frac{C_8}{M}\right)^{m+N} + \sum_{p=0}^{r} q'_p \left(\frac{1}{M}\right)^{m-p} (1+m)^{1+\alpha}\Bigg]. \tag{2.28}$$

Since the quantities $b, \alpha, s_\alpha, c_8, m, q'_p, p$ are indenpedent of N, we can find a sufficiently large $M \ge c_8$ such that

$$\sqrt{e^{(2b/M)^2}-1}\cdot 2^{2+\alpha}\cdot s_\alpha + \left(\frac{C_8}{M}\right)^{m} + \sum_{p=0}^{r} q'_p \left(\frac{1}{M}\right)^{m-p} (1+m)^{1+\alpha} < 1. \tag{2.29}$$

Then

$$\beta^{(1)}_{N+m} \le \frac{(N+m)!}{(N+m)^{1+\alpha}} M^{N+m}.$$

If N (and thus k) is an odd number then we use instead of (2.7) the inequality

$$\int_{R_n} P(x)(|x_1|-|x_2+\ldots+x_n|)^{N-1}|x_1+\ldots+x_n|\,dF_1(x_1)\ldots dF_n(x_n) \leqq \tag{2.30}$$
$$\leqq \sum_{p=0}^{r} |\gamma_p| \int_{R_n} \left(\sum_{j=1}^{n} |x_j|\right)^{N+p} dF_1(x_1)\ldots dF_n(x_n)$$

and we see easily that inequality (2.4) is valid for odd k. This completes the proof of the inequality (2.4).

Hence from (2.4)

$$\lim_{k\to\infty} \left(\frac{\beta_k^{(1)}}{k!}\right)^{1/k} \leqq M \tag{2.31}$$

therefore the characteristic function $f_1(t)$ is regular in the circle of radius $\frac{1}{M}$. It follows that the characteristic function $f_1(z)$ $|z=t+iv,\ t\in R_1,\ v\in R_1|$ is regular at least in the strip $|\mathrm{Im}\, z|<\frac{1}{M}$.

Next we show that the regularity of $f_1(z)$ in a strip $|\mathrm{Im}\, z|<V$ $|V>0|$ implies that $f_1(z)$ is also regular in the strip $|\mathrm{Im}\, z|<V+\frac{1}{M}$. The statement of theorem 2.1. then follows by induction.

Assume that $f_1(z)$ is regular in the strip $|\mathrm{Im}\, z|<V$ and observe that equation (2.1) holds for all complex z in this strip, so that $|x=(x_j)\in R_n|$

$$\int_{R_n} P(x)e^{iz(xa^*)}\,dF_1(x_1)\ldots dF_n(x_n) = \tag{2.32}$$
$$= \sum_{p=0}^{r} \gamma_p \int_{R_n} (xa^*)^p e^{iz(xa^*)}\,dF_1(x_1)\ldots dF_n(x_n).$$

By differentiating (2.32) N-times with respect to z and then putting $z=-iv_0$ we obtain

$$\int_{R_n} P(x)(xa^*)^N e^{v_0(xa^*)}\,dF_1(x_1)\ldots dF_n(x_n) = \tag{2.33}$$
$$= \sum_{p=0}^{r} \gamma_p \int_{R_n} (xa^*)^{N+p} e^{v_0(xa^*)}\,dF_1(x_1)\ldots dF_n(x_n)$$

for $|v_0|<V$. Introduce now the functions

$$G_j(x) = A_j^{-1} \int_{-\infty}^{x} e^{vy}\,dF_j(y) \quad |j=1,2,\ldots,n| \tag{2.34}$$

where

$$A_j = \int_{-\infty}^{\infty} e^{v_0 y}\,dF_j(y) \tag{2.35}$$

Clearly $G_j(x)$ is a distribution function. We denote its characteristic function by $g_j(t)$. We divide both sides of the equation (2.33) by the product $A_1 \dots A_n$ and we see that

$$\int_{R_n} P(x)(xa^*)^N \, dG_1(x_1) \dots dG_n(x_n) = \tag{2.36}$$

$$= \sum_{p=0}^{r} \gamma_p \int_{R_n} (xa^*)^{N+p} \, dG_1(x_1) \dots dG_n(x_n).$$

This relation is identical to (2.2). We proceed in the same way as in the first part of the proof of theorem 2.1., to show that $g_1(z)$ is also an analytic characteristic function which is regular in the strip $|\mathrm{Im}\, z| < \frac{1}{M}$. Since

$$g_1(z) = A_1^{-1} \int_{-\infty}^{\infty} e^{izx + v_0 x} \, dF_1(x),$$

it follows immediately that the characteristic function $f_1(z)$ is regular in the strip $|\mathrm{Im}\, z| < V + \frac{1}{M}$, and so our theorem 2.1. is completely proved.

As extension of theorem 2.1. it is not difficult to prove:

Theorem 2.2. *Let the components of the random vector-variable $\zeta = (\xi_j) \in R_n$ be independent but not necessarily identically distributed random variables. Let the distribution function of ξ_j be denoted by $F_j(x)$ and assume that ξ_j has moments of all orders $|j = 1, 2, \dots, n|$. Let*

$$\Lambda = \zeta\alpha^*, \quad \alpha = (\alpha_j) \in R_n, \quad \alpha_j \neq 0 \quad |j = 1, \dots, n|$$

and let

$$P = \sum A_{\nu_1 \dots \nu_n} \xi_1^{\nu_1} \dots \xi_n^{\nu_n}$$

be a common polynomial statistic of degree p_1 and order m $|m \leqq p_1|$. If P has polynomial regression of order r $|0 \leqq r \leqq m-1|$ on Λ then the characteristic function $f_j(t)$ of $F_j(x)$ is an entire characteristic function $|j = 1, \dots, n|$.

Proof. By the assumption we see, similarly to (2.1), that

$$\int_{R_n} P(x) e^{it \sum_{j=1}^{n} \alpha_j x_j} \, dF_1(x_1) \dots dF_n(x_n) = \tag{2.37}$$

$$= \sum_{p=0}^{r} \gamma_p \int_{R_n} (x\alpha^*)^p e^{it(x\alpha^*)} \, dF_1(x_1) \dots dF_n(x_n).$$

The transformation $y_j = \alpha_j x_j$ $|j = 1, \dots, n|$ yields

$$\int_{R_n} P_1(y) e^{it(ya^*)} \, dG_1(y_1) \dots dG_n(y_n) = \tag{2.38}$$

$$= \sum_{p=0}^{r} \int_{R_n} (ya^*)^p e^{it(ya^*)} \, dG_1(y_1) \dots dG_n(y_n).$$

Here $G_j(x)$ is the distribution function of $\eta_j=\alpha_j\xi_j$ $|j=1, ..., n|$ and

$$P_1(y) = P\left(\frac{y_1}{\alpha_1}, ..., \frac{y_n}{\alpha_n}\right)$$

is also a common polynomial statistic. We conclude in the same way as in the proof of the theorem 2.1., that the characteristic function of $G_j(x)$ and also the characteristic function of $F_j(x)$ are entire functions ($j=1, ..., n$).

Let

$$P = \sum A_{\nu_1 ... \nu_n}\xi_1^{\nu_1} ... \xi_n^{\nu_n}$$

be a polynomial statistic of degree p_1. Suppose that no exponent in P exceeds m while at least one variable has exponent m $|m \leqq p_1|$. We form the polynomial statistic

$$P^* = \frac{1}{n!}\sum{}^* \sum A_{\nu_1 ... \nu_n}\xi_{k_1}^{\nu_1} ... \xi_{k_n}^{\nu_n}$$

where the first summation runs over all permutations $k_1, ..., k_n$ of the first n integers, while the second summation is taken over all subscripts satisfying (1.1). The statistic P^* is said to be the adjoint polynomial statistic of P.

Theorem 2.3. *Let the components of the random vector variable* $\zeta=(\xi_j)\in R_n$ *be independent and identically distributed with* $F(x)$. *Assume that* $F(x)$ *has moments of all orders. Let* $\Lambda=\zeta\alpha^*$, $\alpha=(\alpha_j)\in R_n$, $\alpha_j\neq 0$ *and let*

$$P = \sum A_{\nu_1 ... \nu_n}\xi_1^{\nu_1} ... \xi_n^{\nu_n}$$

be a regular polynomial statistic of degree p_1 *and order* m $|m\leqq p_1|$. *If* P *has polynomial regression of order* r $|0\leqq r\leqq m-1|$ *on* Λ *then* P^* *(the adjoint polynomial statistic of* P*) has polynomial regression of order* r $|0\leqq r\leqq m-1|$ *on* Λ.

Proof. Since P has polynomial regression of order r $|0\leqq r\leqq m-1|$ on Λ, therefore

$$\int_{R_n} P(x)e^{it(x\alpha^*)}\,dF_1(x_1) ... dF_n(x_n) = \tag{2.39}$$
$$= \sum_{p=0}^{r}\gamma_p \int_{R_n}(x\alpha^*)^p e^{it(x\alpha^*)}\,dF_1(x_1) ... dF_n(x_n).$$

By the transformation

$$x_j = \frac{1}{\alpha_j}y_j \quad |j = 1, ..., n|$$

we get the expression (2.38) and then by the transformation $y_j=y_{k_j}$ $|j=1, ..., n|$ the formula

$$\int_{R_n} P(y_{k_1}, ..., y_{k_n})e^{it(ya^*)}\,dG_1(y_{k_1}) ... dG_n(y_{k_n}) = \tag{2.40}$$
$$= \sum_{p=0}^{r}\gamma_p \int_{R_n}(ya^*)^p e^{it(ya^*)}\,dG_1(y_{k_1}) ... dG_n(y_{k_n}),$$

follows, where $k_1, \ldots, k_n$ is a permutation of the first n integers. Now the transformation $y_j=\alpha_j x_j \mid j=1, \ldots, n \mid$ yields

(2.41)
$$\int_{R_n} P(x_{k_1}, \ldots, x_{k_n}) e^{it(x\alpha^*)}\, dF(x_1) \ldots dF(x_n) = \sum_{p=0}^{r} \gamma_p \int_{R_n} (x\alpha^*)^p e^{it(x\alpha^*)}\, dF(x_1) \ldots dF(x_n)$$

i.e. $P(\xi_{k_1}, \ldots, \xi_{k_n})$ and thus P^* has also polynomial regression of order $r \mid 0 \leqq r \leqq m-1 \mid$ on Λ.

Theorem 2.4. *Let the components of $\zeta=(\xi_j)\in R_n$ be independently and identically distributed random variables with distribution function $F(x)$. Assume that $F(x)$ has moments of all orders. Let $\Lambda=\zeta\alpha^*$ where $\alpha=(\alpha_j)\in R_n, \alpha_j \neq 0$ and let*

$$P = \sum A_{\nu_1 \ldots \nu_n} \xi_1^{\nu_1} \ldots \xi_n^{\nu_n}$$

be a polynomial statistic of degree p_1 and order m $\mid m \leqq p_1 \mid$. Suppose that no exponent in P exceeds m while at least one variable has exponent m. Let the adjoint polynomial of P be a nonnegative polynomial. If P has polynomial regression of order $r \mid 0 \leqq r \leqq m-1 \mid$ on Λ then the characteristic function of $F(x)$ is an entire function.

PROOF. The adjoint polynomial P^* of P is nonnegative therefore P^* is a regular polynomial statistic. On the basis of theorem 2.3. we obtain that our theorem 2.4. is a consequence of theorem 2.2.

Acknowledgement: I would like to thank Professor Dr Béla Gyires for his helpful discussion.

References

[1] E. LUKÁCS—R. G. LAHA, Applications of charecteristic functions, *New York*, 1964.

[2] B. GYIRES, On an extension of a theorem of Linnik and Zinger, *Advences in Applied Probability* **7**, (1975), 786—797.

[3] A. A. ZINGER—YU. V. LINNIK, On a class of differential equations with an application to certain question of regression theory, *Selected Translations in Mathematical Statistics and Probability* **2**, (1957), 181—190. *Amer. Math. Soc.*

(Received April 26, 1975.)