On the asymptotic behaviour of the generalized multinomial distributions

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Abstract

In this paper we consider the asymptotic behaviour in weak sence of the generalized multinomial and marginal multinomial distributions. As special cases we obtain from our results the limit theorems of the wellknown multinomial and marginal multinomial distributions. One of them give a possibility to apply the Chisquare method for more general hypothesis as usual.

1. Let R_p be the $p \geqq 2$-dimensional vector space with column vector as its elements. Let

$$A = \begin{pmatrix} a_{11} \dots a_{1p} \\ a_{21} \dots a_{2p} \\ \dots\dots\dots \end{pmatrix}$$

be a stochastic matrix, that is let

$$a_{jk} \geqq 0, \quad \sum_{k=1}^{p} a_{jk} = 1 \quad (j = 1, 2, \dots;\ k = 1, \dots, p).$$

Let A_m be a finite matrix built from the first m rows of A. Let $(A_{\beta_1}^{(1)}, \dots, A_{\beta_p}^{(p)})$ be the matrix, which is built from the columns of A, namely the k-th column of A_m appears β_k-times. If $\beta_k=0$, then the k-th column of A_m is missing from the matrix $(A_{\beta_1}^{(1)}, \dots, A_{\beta_p}^{(p)})$.

Definition 1. The random vector-variable $\eta_m = (\eta_m^{(k)}) \in R_p$ defined on the probability space $(\Omega, \mathscr{A}, P)$ is called a generalized multinomial distributed random vector-variable generated by the matrix A_m, if

$$P\big(\eta_m^{(k)} = \beta_k (k = 1, \dots, p)\big) = \frac{1}{\beta_1! \dots \beta_p!} \operatorname{Per}(A_{\beta_1}^{(1)} \dots A_{\beta_p}^{(p)}), \tag{1}$$

where $\beta_1, \dots, \beta_p$ are non-negative integers which satisfy the condition $\beta_1 + \dots + \beta_p = m$.

Definition 2. The random vector-variable $\eta_m^{(0)} = (\eta_m^{(k)}) \in R_{p-1}$ built from the first $p-1$ components of η_m is called generalized marginal multinomial distributed random vector-variable generated by the matrix A_m.

If all rows of A_m are equal, then η_m and $\eta_m^{(0)}$ are the well-known multinomial and marginal multinomial distribution respectively ([2], 31—32).

The aim of this paper is an investigation of the asymptotic behaviour of the sequences $\{\eta_m\}_{m=1}^{\infty}$ and $\{\eta_m^{(0)}\}_{m=1}^{\infty}$ respectively. In the part 2 we give sufficient conditions for the sequence $\{\eta_m\}_{m=1}^{\infty}$ to converge weakly to a p-dimensional normal distributed random vector-variable. As an application of this theorems in part 3 is proved a theorem, which is suitable for the Chi-square method to apply for more general hypothesis as usual. In part 4 is given a necessary and sufficient condition for the sequence $\{\eta_m^{(0)}\}_{m=1}^{\infty}$ to converge weakly to a $p-1$-variate distribution with independent Poissonian components.

2. We adjoint now to the j-th row of the matrix A the random vector-variable $\xi_j=(\xi_j^{(k)})\in R_p$ defined on the probability space $(\Omega, \mathcal{A}, P)$. Let the conditions

$$\xi_j^{(1)}+\ldots+\xi_j^{(p)}=1,$$

$$P\big(\xi_j^{(k)}=1,\ \xi_j^{(\alpha)}=0\ (\alpha=1,\ldots,p;\ \alpha\neq k)\big)=a_{jk}\quad (k=1,\ldots,p)$$

be satisfied by the components of ξ_j. It is obviously that the characteristic function of ξ_j is equal to

$$a_{j1}e^{it_1}+\ldots+a_{jp}e^{it_p},\quad t=(t_k)\in R_p.$$

Suppose that the elements of the sequence $\{\xi_j\}_{j=1}^{\infty}$ are independent random vector-variables. Thus the characteristic function of the random vector-variable $\xi_1+\ldots+\xi_m$ is equal to

$$\varphi_m(t)=\prod_{j=1}^{m}(a_{j1}e^{it_1}+\ldots+a_{jp}e^{it_p}). \tag{2}$$

On the other hand it was proved by the author ([1], Corollary 1) that (2) is also the characteristic function of the generalized multinomial random vector-variable η_m. Thus

$$\eta_m=\xi_1+\ldots+\xi_m. \tag{3}$$

We obtain easily from (2) that ([1], Corollary 1)

$$E(\eta_m^{(k)})=\sum_{j=1}^{m}a_{jk}\quad (k=1,\ldots,p)$$

and

$$\operatorname{cov}\eta_m=\begin{pmatrix}E(\eta_m^{(1)})\ldots & 0\\ \cdots\cdots\cdots\cdots & \cdots\\ 0 & \ldots E(\eta_m^{(p)})\end{pmatrix}-A_m^*A_m. \tag{4}$$

One can give the following interpretation of the result (3). Let the independent experiments with mutually exclusive and exhaustive events $E_1,\ldots,E_p$ of the probability space $(\Omega,\mathcal{A},P)$ be given. In the j-th experiment let $P(E_k)=a_{jk}$ $(k=1,\ldots,p)$. Then the probability that the event E_k $(k=1,\ldots,p)$ occurs β_k-times in the first m experiments is given by (1), where $\beta_1+\ldots+\beta_p=m$.

Let $r \leqq p$ be the rank of the matrix

$$B_m = \begin{pmatrix} b_{11}^{(m)} \dots b_{1p}^{(m)} \\ \dots\dots\dots \\ b_{r1}^{(m)} \dots b_{rp}^{(m)} \end{pmatrix} \quad (m = 1, 2, \dots)$$

with real elements. Let

$$\zeta_m = (\zeta_m^{(k)}) = B_m(\eta_m - E(\eta_m)) \in R_r \quad (m = 1, 2, \dots),$$

where $\eta_m = (\eta_m^{(k)}) \in R_p$ is the generalized multinomial random vector-variable generated by the matrix A_m. It is obviously that

$$E(\zeta_m) = 0 \in R_r,$$

(5)
$$\Gamma_m = \operatorname{cov} \zeta_m = B_m(\operatorname{cov} \eta_m) B_m^*,$$

$$\operatorname{rank} \Gamma_m \leqq \min \{r, \operatorname{rank} \operatorname{cov} \eta_m\}.$$

Let $g_m(u)$, $u = (u_\beta) \in R_r$ be the characteristic function of ζ_m and let

$$S(B_m) = \sum_{j=1}^{m} \left\{ \left(\sum_{k=1}^{p} a_{jk} h_k(B_m) \right) \left(\sum_{k=1}^{p} a_{jk} h_k^2(B_m) \right) + \sum_{k=1}^{p} a_{jk} h_k^3(B_m) \right\}$$

where

$$h_k(B_m) = \sqrt{\sum_{\beta=1}^{r} (b_{\beta k}^{(m)})^2} \quad (k = 1, \dots, p).$$

Theorem 1. *If*

(6)
$$\lim_{m \to \infty} S(B_m) = 0,$$

then

$$\lim_{m \to \infty} g_m(u) e^{\frac{1}{2} u^* \Gamma_m u} = 1, \quad u \in R_r.$$

PROOF. By the application of the notation

$$b_k^{(m)} = \sum_{\beta=1}^{r} b_{\beta k}^{(m)} u_\beta \quad (k = 1, \dots, p)$$

in the characteristic function

$$g_m(u) = \exp\{-iu^* B_m E(\eta_m)\} \cdot E(\exp\{iu^* B_m \eta_m\}),$$

we get

(7)
$$u^* B_m E(\eta_m) = \sum_{j=1}^{m} \sum_{k=1}^{p} a_{jk} b_k^{(m)}, \quad u^* B_m \eta_m = \sum_{k=1}^{p} b_k^{(m)} \eta_m^{(k)}.$$

Therefore

$$g_m(u) = \exp\{-iu^* B_m E(\eta_m)\} \cdot \varphi_m(b_1^{(m)}, \dots, b_p^{(m)}),$$

where the function $\varphi_m(t)$ is defined by (2). From the last expression

$$\log g_m(u) = -iu^* B_m(\eta_m) + \sum_{j=1}^{m} \log (a_{j1} e^{ib_1^{(m)}} + \dots + a_{jp} e^{ib_p^{(m)}}).$$

By application of the Taylor formula we get

$$\log g_m(u) = -iu^* B_m E(\eta_m) +$$
$$+ \sum_{j=1}^{m} \log\left[1 + i \sum_{k=1}^{p} a_{jk} b_k^{(m)} - \frac{1}{2} \sum_{k=1}^{p} a_{jk} (b_k^{(m)})^2 + O\left(\sum_{k=1}^{p} a_{jk} (b_k^{(m)})^3\right)\right].$$

We expend the logarithm and use (7) to obtain

$$\log g_m(u) + \frac{1}{2} \sum_{j=1}^{m} \left\{ \sum_{k=1}^{p} a_{jk} (b_k^{(m)})^2 - \left(\sum_{k=1}^{p} a_{jk} b_k^{(m)} \right)^2 \right\} = O\left(\sum_{j=1}^{m} S_j \right),$$

where

$$S_j = \left(\sum_{k=1}^{p} a_{jk} b_k^{(m)} \right) \left(\sum_{k=1}^{p} a_{jk} (b_k^{(m)})^2 \right) + \sum_{k=1}^{p} a_{jk} (b_k^{(m)})^3 \quad (j = 1, \ldots, m).$$

Since

$$|b_k^{(m)}| \leqq h_k(B_m)(u^*u)^{1/2},$$

therefore

$$\sum_{j=1}^{m} |S_j| \leqq S(B_m)(u^*u)^{3/2}.$$

According to the assumption (6) and the expressions (4) and (5),

$$\lim_{m\to\infty} \left\{ \log g_m(u) + \frac{1}{2} u^* \Gamma_m u \right\} = 0$$

and this is the statement of our Theorem 1.

As an important special case of the Theorem 1 is the following Theorem:

Theorem 2. *Let* $B_m = \left(\frac{1}{m}\right)^{\alpha} C_m$, $\alpha > \frac{1}{3}$ *and let the elements of the matrix-sequence* $\{C_m\}_{m=1}^{\infty}$ *bounded. Then*

$$\lim_{m\to\infty} g_m(u) e^{\frac{1}{2} u^* \Gamma_m u} = 1, \quad u \in R_r.$$

Proof. Let $C = (c_{\beta k}^{(m)})$. Since $\{C_m\}_{m=1}^{\infty}$ is bounded, we see that $\frac{1}{m} S(C_m)$ is also bounded, hence

$$S(B_m) = \left(\frac{1}{m}\right)^{3\alpha} S(C_m) = O\left(\left(\frac{1}{m}\right)^{3\alpha - 1} \right),$$

that is the assumption (6) is satisfied.

If $C_m = B$ $(m = 1, 2, \ldots)$, then the sequence $\{C_m\}_{m=1}^{\infty}$ is bounded obviously. We obtain therefore the following Corollary:

Corollary 1. If

$$B_m = \left(\frac{1}{m}\right)^{\alpha} B (m = 1, 2, \ldots), \quad \alpha > \frac{1}{3}, \quad \operatorname{rank} B = r \leqq p,$$

then

$$\lim_{m\to\infty} g_m(u) e^{\frac{1}{2} u^* \Gamma_m u} = 1, \quad u \in R_r.$$

Theorem 3. *Let* $B_m = \frac{1}{\sqrt{m}} C_m$. *Assume that the limits*

$$\lim_{m\to\infty} C_m = B = (b_{jk}), \quad \text{rank } B = r,$$

$$\lim_{j\to\infty} a_{jk} = a_k \quad (k = 1, \dots, p), \quad \sum_{k=1}^{p} a_k = 1 \tag{8}$$

exist. Then

$$\lim_{m\to\infty} g_m(u) = e^{-\frac{1}{2} u^* \Gamma u}, \quad u \in R_r, \tag{9}$$

where $\Gamma = BGB^*$, *with*

$$G = \begin{pmatrix} a_1 & \dots & (0) \\ \cdots & \cdots & \cdots \\ (0) & \dots & a_p \end{pmatrix} - aa^*, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}.$$

PROOF. Using (8) we get

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} a_{jk} = a_k, \quad \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} a_{jk} a_{jl} = a_k a_l.$$

Applying the notation

$$b_k = \sum_{\beta=1}^{r} b_{\beta k} u_\beta \quad (k = 1, \dots, p)$$

the relations

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \sum_{k=1}^{p} a_{jk} (b_k^{(m)})^2 = \sum_{k=1}^{p} a_k b_k^2,$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{p} a_{jk} b_k^{(m)} \right)^2 = \left(\sum_{k=1}^{p} a_k b_k \right)^2$$

hold and thus in according to our statement

$$\lim_{m\to\infty} u^* \Gamma_m u = \sum_{k=1}^{p} a_k b_k^2 - \left(\sum_{k=1}^{p} a_k b_k \right)^2 = u^* \Gamma u.$$

As a consequence of the Theorem 3, we obtain following in the literature well-known Corollary:

Corollary 2. If

$$B_m = \frac{1}{\sqrt{m}} B (m = 1, 2, \dots), \quad \text{rank } B = r$$

and

$$a_{jk} = a_k \quad (j = 1, 2, \dots;\ k = 1, \dots, p),$$

then the limit (9) holds.

3. We consider now the following important special case of Theorem 2.

Theorem 4. *Suppose that the elements of the matrix A satisfy the conditions*

$$0 < b \leqq a_{jk} \quad (j = 1, 2, \ldots; \; k = 1, \ldots, p). \tag{10}$$

Let $B_m = \frac{1}{\sqrt{m}} C_m$, *where*

$$C_m = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1^{(m)}}} & \cdots & (0) \\ \cdots & \cdots & \cdots \\ (0) & \cdots & \frac{1}{\sqrt{\lambda_p^{(m)}}} \end{pmatrix} \quad (m = 1, 2, \ldots) \tag{11}$$

with

$$\lambda_k^{(m)} = \frac{1}{m} \sum_{j=1}^{m} a_{jk} \quad (m = 1, 2, \ldots; \; k = 1, \ldots, p).$$

Then

$$\lim_{m \to \infty} g_m(u) e^{\frac{1}{2} u^* \Gamma_m u} = 1, \quad u \in R_p$$

with

$$\Gamma_m = I - \left(\frac{\sum_{j=1}^{m} a_{jk} a_{jl}}{\sqrt{\sum_{j=1}^{m} a_{jk} \sum_{j=1}^{m} a_{jl}}} \right)_{k, l=1}^{p},$$

where J is the unit matrix of p-th order.

Proof. It follows from the assumption (10) that the sequence $\{C_m\}_{m=1}^{\infty}$ is bounded, thus the Theorem 2 is applicable. We can derive the expression of Γ_m easily from (4) and (5) respectively.

Theorem 5. *Let the elements of the matrix A be positive numbers and let*

$$\lim_{j \to \infty} a_{jk} = a_k > 0 \quad (k = 1, \ldots, p). \tag{12}$$

If $B_m = \frac{1}{\sqrt{m}} C_m$ *and the sequence* $\{C_m\}_{m=1}^{\infty}$ *is defined by (11), then*

$$\lim_{m \to \infty} g_m(u) = e^{-\frac{1}{2} u^* \Gamma u}, \quad u \in R_p \tag{13}$$

with $\Gamma = J - bb^*$, *where the components of the vector* $b \in R_p$ *are one after another* $\sqrt{a_k}$ $(k = 1, \ldots, p)$ *and* rank $\Gamma = p - 1$.

Proof. It follows from the assumption of the Theorem 5 that the condition (10) is satisfied, thus the Theorem 4 is applicable. If we use (11) and we take in consideration that the limit (12) holds, then in accordance to our statement

$$\lim_{m \to \infty} u^* \Gamma_m u = u^* \left(\delta_{kl} - \sqrt{a_k a_l} \right)_{k, l=1}^{p} u,$$

where δ_{kl} is the Kronecker symbol.

The following special case of Theorem 5 is a well-known statement in the literatur.

Corollary 3. Suppose that the elements of the matrix A satisfy the conditions

$$a_{jk} = a_k > 0 \quad (j = 1, 2, \dots;\ k = 1, \dots, p).$$

Let $B_m = \frac{1}{\sqrt{m}} C_m$, where

$$C_m = \begin{pmatrix} \frac{1}{\sqrt{a_1}} & \dots & (0) \\ \dots & \dots & \dots \\ (0) & \dots & \frac{1}{\sqrt{a_p}} \end{pmatrix} \quad (m = 1, 2, \dots).$$

Then the limit (13) holds.

We can express Theorem 5 in the following form also:

Let the elements of the matrix A be positive numbers, which satisfy the condition (12). Let $B_m = \frac{1}{\sqrt{m}} C_m$ and let the elements of the sequence $\{C_m\}_{m=1}^{\infty}$ be defined by (11). Then

$$B_m(\eta_m - E(\eta_m)) \Rightarrow N(0, I - bb^*), \quad m \to \infty$$

and the rank of the matrix $J - bb^*$ is equal to $p-1$.

If we use Theorem 3.4.2. of the monograph [2], we obtain the following result:

Theorem 6. *Let the elements of the matrix A be positive numbers which satisfy the conditions (12). Let $B_m = \frac{1}{\sqrt{m}} C_m$ and let the elements of the sequence $\{C_m\}_{m=1}^{\infty}$ be defined by (11). Then the sequence of the random variables*

$$\left\{ \sum_{k=1}^{p} \frac{\left(\eta_m^{(k)} - \sum_{j=1}^{m} a_{jk} \right)^2}{\sum_{j=1}^{m} a_{jk}} \right\}_{m=1}^{\infty}$$

converges weakly to the Chi-square distribution with degrees of freedom $p-1$.

This Theorem give a possibility to generalize the Chisquare test in the following way:

Let the independent experiments with mutually exclusive and exhaustive events $E_1, \dots, E_p$ of the probability space $(\Omega, \mathcal{A}, P)$ be given. Then the H_0 hipothesis is the following:

In the j-th experiment

$$P(E_k) = a_{jk} > 0 \quad (k = 1, 2, \dots, p;\ j = 1, 2, \dots)$$

and

$$\lim_{j \to \infty} a_{jk} = a_k > 0 \quad (k = 1, \dots, p).$$

4. Assume that the matrix-sequence $\{A_m\}_{m=1}^{\infty}$ satisfies the conditions

$$A_m = (a_{jk}^{(m)}), \quad a_{jk}^{(m)} \geqq 0, \quad \sum_{k=1}^{p} a_{jk}^{(m)} = 1$$

$$(k = 1, \dots, p; \; j = 1, \dots, m; \; m = 1, 2, \dots).$$

Let $\eta_m \in R_p$ be the generalized multinomial random vector variable generated by the matrix A_m. We see from (2) that the corresponding characteristic function is equal to

$$\varphi_m(t) = \prod_{j=1}^{m} (a_{j1}^{(m)} e^{it_1} + \dots + a_{jp}^{(m)} e^{it_p}), \quad t = (t_k) \in R_p. \tag{14}$$

Suppose that the random vector-variable

$$\begin{array}{l} \xi_{11} \\ \xi_{21}\; \xi_{22} \\ \cdot \quad \cdot \\ \xi_{m1}\, \xi_{m2} \dots \xi_{mm} \\ \cdot \quad \cdot \quad \dots \quad \cdot \end{array}$$

satisfy the following conditions: $\xi_{mj} = (\xi_{mj}^{(k)}) \in R_p$ and the random variable $\xi_{mj}^{(k)}$ has the values 0, 1, namely

$$P\big(\xi_{mj}^{(k)} = 1, \quad \xi_{mj}^{(\alpha)} = 0 \quad (\alpha = 1, \dots, p; \; \alpha \neq k)\big) = a_{jk}^{(m)}$$

$$(k = 1, \dots, p; \; j = 1, \dots, m; \; m = 1, 2, \dots)$$

and the random vector-variables belong to the same row are independent. Then on accordance to (3)

$$\eta_m = \xi_{m1} + \dots + \xi_{mm}.$$

Let $\eta_m^{(0)} \in R_{p-1}$ be the generalized marginal multinomial random vector-variable generated by the matrix A_m. Substituting $t_p = 0$ in (14) we get the characteristic function $\varphi_m^{(0)}(t)$, $t = (t_k) \in R_{p-1}$ of the random vector-variable $\eta_m^{(0)}$. Thus

$$\varphi_m^{(0)}(t) = \prod_{j=1}^{m} [1 + a_{j1}^{(m)}(e^{it_1} - 1) + \dots + a_{jp-1}^{(m)}(e^{it_{p-1}} - 1)]. \tag{15}$$

Theorem 7. *The sequence of the random vector-variables* $\{\eta_m^{(0)}\}_{m=1}^{\infty}$ *converges weakly if and only if to a* $p-1$ *variate distribution with independent Poissonian components, when the conditions*

$$\lim_{m \to \infty} \sum_{j=1}^{m} a_{jk}^{(m)} = \lambda_k \quad (k = 1, \dots, p-1), \tag{16}$$

$$\lim_{m \to \infty} \sum_{j=1}^{m} (1 - a_{jp}^{(m)})^2 = 0 \tag{17}$$

are satisfied.

PROOF. It follows from (15) that

$$\log \varphi_m^{(0)}(t) = \sum_{k=1}^{p-1} (e^{it_k}-1) \sum_{j=1}^{m} a_{jk}^{(m)} + O(S_m(t)), \tag{18}$$

where

$$S_m(t) = \sum_{j=1}^{m} [a_{j1}^{(m)}(e^{it_1}-1)+\ldots+a_{jp-1}^{(m)}(e^{it_{p-1}}-1)]^2.$$

In consideration of

$$|S_m(t)| = \left|\sum_{\alpha=1}^{p-1}\sum_{\beta=1}^{p-1}(e^{it_\alpha}-1)(e^{it_\beta}-1)\sum_{j=1}^{m} a_{j\alpha}^{(m)} a_{j\beta}^{(m)}\right| \leqq 4 \sum_{j=1}^{m}(1-a_{jp}^{(m)})^2, \tag{19}$$

and since in (19) is an equality if

$$t_\alpha = (2k_\alpha+1)\pi \quad (\alpha = 1, \ldots, p-1)$$

with arbitrary integers k_α, we see from (18) that the conditions (16) and (17) are necessary and sufficient to the existence of the relation

$$\lim_{m\to\infty} \varphi_m^{(0)}(t) = \exp\{\lambda_1(e^{it_1}-1)+\ldots+\lambda_{p-1}(e^{it_{p-1}}-1)\}.$$

If

$$a_{jk}^{(m)} = a_k^{(m)}, \quad ma_k^{(m)} = \lambda_k > 0 \quad (k = 1, \ldots, p-1;\ m = 1, 2, \ldots), \tag{20}$$

then simultaneously

$$a_{jp}^{(m)} = 1 - \frac{\lambda_1+\ldots+\lambda_{p-1}}{m} \quad (j = 1, \ldots, m),$$

thus

$$\sum_{j=1}^{m}(1-a_{jp}^{(m)})^2 = \frac{(\lambda_1+\ldots+\lambda_{p-1})^2}{m} \quad (j = 1, \ldots, m),$$

that is the conditions (16) and (17) are satisfied.

We use now Theorem 7 to obtain the following well-konown result:

Corollary 4. Let $\eta_m^{(0)} \in R_{p-1}$ be the generalized marginal multinomial random vector-variable generated by the matrix A_m. If the sequence $\{A_m\}_{m=1}^\infty$ satisfies the condition (20) then the sequence $\{\eta_m^{(0)}\}_{m=1}^\infty$ converges weakly to a $p-1$ variate distribution with independent Poissonian components.

References

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