Functional equations on ordered fields

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0. Introduction

Let G be a multiplicative Abelian group and let A be an additive Abelian group. Let $f: G \to A$ be an arbitrary function and denote

$$\Delta_y f(x) = f(xy) - f(x) \quad (x \in G), \tag{1}$$

where y is a fixed element of G. N. G. DE BRUIJN [2] was the first to deal with the following type of problem: Let H be a set of functions $g: G \to A$ having a given property. If $f: G \to A$ is such a function that $\Delta_y f: G \to A$ belongs to H for every $y \in G$ then one can ask whether there exists $f^* \in H$ such that $f - f^*$ is a homomorphism of G into A. In the affirmative case the property which defines the set H is said to be a *difference-property*. N. G. de Bruijn's investigations [2] are based upon the Haar-measure defined on topological groups (see F. W. CARROLL [3] and F. W. CARROLL—F. S. KOEHL [4]) hence they are said to be analytic investigations. At the same time to our knowledge the algebraic character of this problem has not been investigated yet. Presumably the reason is the difficulty to characterize a difference-property in a purely algebraic way. In this paper we give such a property where the group G is the multiplicative group of the positive elements of an ordered field. With the help of our result we give the general solution of Vajzovič's functional equation of which only the measurable solution has been known so far (see F. VAJZOVIČ [8], K. LAJKÓ [7]).

1. Jensen-property

Let F be an ordered field and denote by P the set of positive elements in F. Furthermore let A be an additive Abelian group.

The function $\alpha: P \to A$ is a *Jensen-function* if

$$2\alpha\left(\frac{x+y}{2}\right) = \alpha(x) + \alpha(y) \tag{2}$$

is valid for all $x, y \in P$. Denote $J(P \to A)$ the class of Jensen-functions.

Theorem 1. *Let $f:P\to A$ be an arbitrary function for which the following property is valid: The function $\Delta_y f(x)=f(xy)-f(x)$ $(x\in P)$ is a Jensen-function for every fixed $y\in P$. Then there exists a function $\alpha\in J(P\to A)$ such that the function $m(x)=$ $=f(x)-\alpha(x)$ $(x\in P)$ is a homomorphism, i.e. $m(xy)=m(x)+m(y)$ for all $x, y\in P$.*

PROOF. (i) If $\beta\in J(P\to A)$ then

(3) $$\beta(2x)-2\beta(x) = \beta(2)-2\beta(1) \quad (x\in P).$$

Indeed from (2) we have

(4) $$\beta(2x) = 2\beta(x+1)-\beta(2) \quad (x\in P).$$

On the other hand from (2) it follows that

$$\beta(x+1)+\beta(1) = 2\beta\left(\frac{x}{2}+1\right)$$

and

$$\beta(x)+\beta(2) = 2\beta\left(\frac{x}{2}+1\right),$$

and so we have

(5) $$\beta(x+1)-\beta(x) = \beta(2)-\beta(1) \quad (x\in P).$$

Using the equations (4) and (5)

$$\beta(2x)-2\beta(x) = 2\beta(x+1)-\beta(2)-2\beta(x) = 2[\beta(2)-\beta(1)]-\beta(2) = \beta(2)-2\beta(1),$$

i.e. (3) is valid.

(ii) Let $y\in P$ be arbitrary. Then $\Delta_y f:P\to A$ belongs to $J(P\to A)$ and so because of (3) we have

(6) $$\Delta_y f(2x)-2\Delta_y f(x) = \Delta_y f(2)-2\Delta_y f(1) = \gamma(y)$$

where $\gamma:P\to A$ is an unknown function. On the other hand with the notation

$$\varphi(x) = f(2x)-2f(x) \quad (x\in P)$$

we get

(7) $$\Delta_y f(2x)-2\Delta_y f(x) = f(2xy)-f(2x)-2f(xy)+2f(x) = \varphi(xy)-\varphi(x)$$

and so because of (6) there follows

(8) $$\varphi(xy) = \varphi(x)+\gamma(y)$$

for all $x, y\in P$. Now let $x=1$ in (8) and let $c=\varphi(1)\in A$, then

$$\gamma(y) = \varphi(y)-c$$

for all $y\in P$. Substituting this into (8) one gets

$$\varphi(xy) = \varphi(x)+\varphi(y)-c$$

for all $x, y\in P$. Putting

$$m(x) = \varphi(x)-c \quad (x\in P)$$

we see that $m:P\to A$ is a homomorphism.

By hypothesis

$$\alpha(x) = f(2x)-f(x)+c \quad (x\in P)$$

belongs to $J(P\to A)$. Hence

$$\alpha(x)-f(x) = f(2x)-2f(x)-c = \varphi(x)-c = m(x)$$

i.e. $\alpha-f$ is a homomorphism. Therefore it follows obviously that $f-\alpha$ is a homomorphism too.

Remark. Theorem 1. can be formulated according to the terminology mentioned in the Introduction in the following way: *If P is the multiplicative group of positive elements of an ordered field then the Jensen-property is a difference-property.*

Here the function $\alpha: P\to A$ has the Jensen-property if $\alpha\in J(P\to A)$.

2. On the Vajzovič-equation

Let $F=\mathbf{R}$ where $\mathbf{R}$ is the (ordered) field of the real numbers. Then $P=\mathbf{R}_+$ where $\mathbf{R}_+$ is the set of positive real numbers. F. Vajzovič [8] dealt with the following problem: Let $f:\mathbf{R}_+\to\mathbf{R}$ be an unknown function, which satisfies the functional equation

$$2f\left(\frac{t}{2}\right) = f\left(\frac{tc}{1+c}\right)+f\left(\frac{t}{1+c}\right)-f\left(\frac{c}{1+c}\right)-f\left(\frac{1}{1+c}\right) \tag{9}$$

for all $t, c\in\mathbf{R}_+$. F. Vajzovič [8] proved that if $f:\mathbf{R}_+\to\mathbf{R}$ is measurable and satisfies (9) then it has the form

$$f(x) = a\log x+bx+a\log 2-\frac{b}{2} \quad (x\in\mathbf{R}_+), \tag{10}$$

where $a, b\in\mathbf{R}$ are arbitrary constants. For this result K. Lajkó [7] gave an elementary proof based on the following lemma which he established in the case $P=\mathbf{R}_+$ and $A=\mathbf{R}$.

Lemma 1. *Let P be the set of positive elements of an ordered field and let A be an additive Abelian group. Let $f: P\to A$ be an unknown function which satisfies the functional equation* (9) *for all $t, c\in P$. Then the function $\Delta_y f(x)=f(xy)-f(x)$ $(x\in P)$ is a Jensen-function for all fixed $y\in P$.*

Proof. Let $x_1, x_2\in P$ arbitrary and

$$t = x_1+x_2, \quad c = \frac{x_1}{x_2}. \tag{11}$$

In (11) t and c belong to P, therefore from (9) we infer for the unknown function $f: P\to A$ the functional equation

$$2f\left(\frac{x_1+x_2}{2}\right) = f(x_1)+f(x_2)-f\left(\frac{x_1}{x_1+x_2}\right)-f\left(\frac{x_2}{x_1+x_2}\right) \tag{12}$$

for all $x_1, x_2 \in P$. Now let $y \in P$ fixed then (12) implies

$$2\Delta_y f\left(\frac{x_1+x_2}{2}\right) = 2f\left(\frac{x_1+x_2}{2}\,y\right) - 2f\left(\frac{x_1+x_2}{2}\right) =$$

$$= f(x_1y)+f(x_2y)-f\left(\frac{x_1y}{x_1y+x_2y}\right)-f\left(\frac{x_2y}{x_1y+x_2y}\right)-$$

$$-f(x_1)-f(x_2)+f\left(\frac{x_1}{x_1+x_2}\right)+f\left(\frac{x_2}{x_1+x_2}\right) = \Delta_y f(x_1)+\Delta_y f(x_2)$$

i.e. $\Delta_y f \in J(P \to A)$. Hence the lemma is proved.

From this lemma there follows the following

Theorem 2. *Let P be the set of positive elements of an ordered field and let A be an additive Abelian group. Let $f: P \to A$ be an unknown function which satisfies the functional equation* (9) *for all $t, c \in P$. Then there exist functions $\alpha: P \to A$ and $m: P \to A$ such that*

$$f(x) = \alpha(x)+m(x) \quad (x \in P) \tag{13}$$

holds, where $\alpha \in J(P \to A)$ and $m: P \to A$ is a homomorphism with $2\left[\alpha\left(\frac{1}{2}\right)+m\left(\frac{1}{2}\right)\right]=0$.

Proof. By the lemma $\Delta_y f \in J(P \to A)$ for arbitrary $y \in P$. Hence on the basis of Theorem 1. there exists $\alpha \in J(P \to A)$ such that $m = f - \alpha$ is a homomorphism, i.e. (13) holds, where α is a Jensen-function and m is a homomorphism. Substituting (13) into (9) we get that $2\left[\alpha\left(\frac{1}{2}\right)+m\left(\frac{1}{2}\right)\right]=0$. Hence the theorem 2. is proved.

In case $P = \mathbf{R}_+$ we can give the general form of Jensen-functions by means of the general solution of the Cauchy-functional equation

$$A(x+y) = A(x)+A(y), \tag{14}$$

where $x, y \in \mathbf{R}$.

There holds the following

Lemma 2. *If the function $\alpha: \mathbf{R}_+ \to \mathbf{R}$ satisfies the Jensen-functional equation* (2) *for all $x, y \in \mathbf{R}_+$, then α has the form*

$$\alpha(x) = A(x)+C \quad (x \in \mathbf{R}_+), \tag{15}$$

where A satisfies the functional equation (14) *for all $x, y \in \mathbf{R}$ and C is an arbitrary constant.*

Proof. Putting $x \to x+x_0$, $y \to y+x_0$ in (2) we get

$$2\alpha\left(\frac{x+y}{2}+x_0\right) = \alpha(x+x_0)+\alpha(y+x_0), \quad x, y > -x_0. \tag{16}$$

Putting $x \to x+y$, $y \to 0$ in (16) we obtain

$$(17) \qquad 2\alpha\left(\frac{x+y}{2}+x_0\right) = \alpha(x+y+x_0)+\alpha(x_0), \quad x+y > -x_0.$$

From (16) and (17) we obtain the functional equation

$$(18) \qquad \alpha(x+y+x_0)+\alpha(x_0) = \alpha(x+x_0)+\alpha(y+x_0)$$

for all $(x, y)\in D=\{(x, y)|x, y>-x_0, x+y>-x_0\}$. From (18) we infer that the function $\bar{A}$ defined by

$$(19) \qquad \bar{A}(x) = \alpha(x+x_0)-\alpha(x_0); \quad \bar{A}:(-x_0, \infty) \to \mathbf{R}$$

is additive on D. Hence from results of Z. Daróczy—L. Losonczi [5] there follows that $\bar{A}$ has one and only one extension A which is additive on $\mathbf{R}$ and $\bar{A}(x)=A(x)$ for all $x\in(-x_0, \infty)$ and so from (19) we obtain that

$$\alpha(x) = A(x-x_0)+\alpha(x_0) \quad (x\in\mathbf{R}_+)$$

which gives (15) with $C=\alpha(x_0)-A(x_0)$.

Now it is easy to prove the following

Theorem 3. *Let $f:\mathbf{R}_+\to\mathbf{R}$ be an unknown function which satisfies the functional equation* (9) *for all $t, c\in\mathbf{R}_+$ then f has the form*

$$(20) \qquad f(x) = A\left(x-\frac{1}{2}\right)+m(2x) \quad (x\in\mathbf{R}_+),$$

where A satisfies the functional equation (14) for all $x, y\in\mathbf{R}$ and m satisfies the functional equation

$$(21) \qquad m(xy) = m(x)+m(y) \quad (x, y\in\mathbf{R}_+).$$

Proof. Using theorem 2. in case $P=\mathbf{R}_+$ and lemma 2. we have

$$f(x) = A(x)+m(x)+C \quad (x\in\mathbf{R}_+),$$

where $A\left(\frac{1}{2}\right)+C+m\left(\frac{1}{2}\right)=0$, that is $C=-A\left(\frac{1}{2}\right)-m\left(\frac{1}{2}\right)$ and so (20) follows.

From our general result there follows

Theorem 4. *Let $f:\mathbf{R}_+\to\mathbf{R}$ be an unknown function which satisfies the functional equation* (9) *for all $t, c\in\mathbf{R}_+$. In this case the following conditions are equivalent:*

a) *There exists $x_0\in\mathbf{R}_+$ such that f is continuous at the points x_0 and $2x_0$;*

b) *The function f is measurable;*

c) *There exists $\varepsilon\in\mathbf{R}_+$ such that f is bounded on the interval $(0, \varepsilon)$.*

*Further from any of these conditions there follows that f has the form (*10*), where a and b are constants.*

PROOF. It will be sufficient to show that f has the form (10) in all three cases.

a) We obtain from theorem 3.

$$f(2x)-f(x) = A\left(2x-\frac{1}{2}\right)+m(4x)-A\left(x-\frac{1}{2}\right)-m(2x) = A(x)+m(2). \tag{22}$$

If f is continuous at the points $x_0, 2x_0$ then the function A is continuous at the point x_0 too, thus A is continuous on $\mathbf{R}$ (see [1]), and so

$$A(x) = bx \quad (x\in\mathbf{R}_+) \tag{23}$$

follows.

On the other hand

$$\begin{aligned} f(2x)-2f(x) &= A\left(2x-\frac{1}{2}\right)+m(4x)-2A\left(x-\frac{1}{2}\right)-2m(2x) = \\ &= A\left(\frac{1}{2}\right)-m(2)-m(x) \end{aligned} \tag{24}$$

holds. From (24) there follows that m is continuous at x_0 and therefore f is continuous on $\mathbf{R}_+$ (see [1]) and so m has the form

$$m(x) = a\log x \quad (x\in\mathbf{R}_+). \tag{25}$$

Using (23) and (25) there follows from (20) that f has the form (10).

b) If f is measurable then there follows from (22) and (24) that the functions A and m are measurable too, thus A and m are of the form (23) and (25) respectively (see [1]), i.e. f has the form (10).

c) If f is bounded on $(0, \varepsilon)$ then there follows from (22) that A is bounded on $\left(0, \frac{\varepsilon}{2}\right)$ and so A has the form (23) (see [1]).

From (24) there follows that m is bounded on $(0, \varepsilon)$ and so m has the form (25) (see [1]). Thus f has the form (10).

3. Further applications

Let P be the set of positive elements of an ordered field and let A be an additive Abelian group. Let $\Delta: P^2\to A$ satisfy the following equations:

$$\Delta(x, y) = \Delta(y, x) \quad (x, y\in P), \tag{26}$$

$$\Delta(x, y)+\Delta(xy, z) = \Delta(x, yz)+\Delta(y, z) \quad (x, y, z\in P), \tag{27}$$

$$2\Delta\left(\frac{x+y}{2}, z\right) = \Delta(x, z)+\Delta(y, z) \quad (x, y, z\in P). \tag{28}$$

The following theorem is valid:

Theorem 5. *Let A be a divisible Abelian group. If $\Delta: P^2 \to A$ satisfies the functional equations* (26), (27) *and* (28), *then there exists a function $\alpha \in J(P \to A)$ such that*

$$\Delta(x, y) = \alpha(xy) - \alpha(x) - \alpha(y) \tag{29}$$

holds for all $x, y \in P$.

PROOF. If $\Delta: P^2 \to A$ satisfies equations (26) and (27), then by the divisible property of A, according to the result of JESSEN—KARPF—THORUP [6], there exists a function $f: P \to A$ such that

$$\Delta(x, y) = f(xy) - f(x) - f(y) \tag{30}$$

holds for all $x, y \in P$. On the other hand because of (28) we obtain from (30) for all fixed $y \in P$ and for arbitrary $x_1, x_2 \in P$

$$2\Delta_y f\left(\frac{x_1+x_2}{2}\right) = 2\Delta\left(\frac{x_1+x_2}{2}, y\right) + 2f(y) =$$

$$= \Delta(x_1, y) + \Delta(x_2, y) + 2f(y) =$$

$$= f(x_1 y) - f(x_1) - f(y) + f(x_2 y) - f(x_2) - f(y) + 2f(y) = \Delta_y f(x_1) + \Delta_y f(x_2),$$

that is $\Delta_y f(x)$ $(x \in P)$ is a Jensen-function. Then according to theorem 1. there exists a function $\alpha \in J(P \to A)$ such that $m(x) = f(x) - \alpha(x)$ is a homomorphism. But from (30) we have

$$\Delta(x, y) = f(xy) - f(x) - f(y) = \alpha(xy) + m(xy) - \alpha(x) - m(x) - \alpha(y) - m(y) =$$

$$= \alpha(xy) - \alpha(x) - \alpha(y),$$

thus the theorem is proved.

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