Remarks on the paper "Equivalent topological properties of the space of signatures of a semilocal ring" by A. Rosenberg and R. Ware

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Dedicated to the memory of Andor Kertész

Rosenberg and Ware deal in their paper [8] with semilocal rings in which 2 is a unit. I observed that the assumption that 2 is a unit can be removed in their results by adding further ideas to their proofs. Theorem 2.1 in [8] can even be generalized to semilocal rings with involutions. The interested reader should understand in detail the paper [8] of Rosenberg—Ware before studying the present paper, since the main ideas are developed there.

We essentially use the same notations as in [8]. Let A be an arbitrary connected semilocal ring (without involution) and let E be a symmetric bilinear space over A. For every signature σ of A we have $|\sigma(E)| \leqq \dim E$ and $\sigma(E) \equiv \dim E \bmod 2$. As in [5] we call E *positive definite* at the signature σ (resp. negative definite, resp. definite), if $\sigma(E) = \dim E$ (resp. $\sigma(E) = -\dim E$, resp. $|\sigma(E)| = \dim E$), and we call E *indefinite* at σ if $|\sigma(E)| < \dim E$. Notice that if $\alpha\colon A \to R$ is a homomorphism from A into a real closed field R inducing σ then E is positive definite (negative definite, ...) at σ if and only if the bilinear space $E \otimes_A R$ is positive definite (negative definite, ...) over R in the classical sense. Such homomorphisms α always exist [4, § 4]. If E is a quadratic space over A, i.e. a projective module equipped with a quadratic form $q\colon E \to A$, such that the associated bilinear form is nondegenerate [cf. 7, p. 110], then we denote the associated bilinear space by $\tilde{E}$. For any signature σ of A we shortly write $\sigma(E)$ instead of $\sigma(\tilde{E})$, and we call E positive definite (negative definite, ...) if $\tilde{E}$ has this property.

Lemma 1. *Let E be a quadratic space which is positive definite at σ. Then $\sigma(c) = 1$ for every unit c of A represented by E.*

Proof. Suppose that $\sigma(c) = -1$. Then the bilinear space $F := \tilde{E} \perp \langle -c \rangle$ is positive definite at σ. Now F represents the element $2c - c = c$. Thus $\sigma(c) = +1$ [6, Lemma 2.3], a contradiction.

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We choose a natural number $h \geqq 1$ such that both $2h-1$ and $4h-1$ are units in A, which is always possible [5, p. 49], and introduce the binary quadratic space G with basis u, v over A and quadratic form

$$q(\lambda u+\mu v) = \lambda^2+\lambda\mu+\mu^2 h.$$

As in [5] this space G will play a useful role in our present paper.

Lemma 2. *$\sigma(G)=2$ for every signature σ of A.*

PROOF. As observed in [5, p. 55f]

$$\tilde{G} \perp \langle -1\rangle \cong \langle 1, 2h-1, (2h-1)(1-4h)\rangle.$$

Apply σ to this relation.

We now turn to Rosenberg—Ware's first theorem [8, Th. 2.2]. We do not need to discuss the statements (3)—(5) in this theorem, since this is already done there [7, Remark 2.3]. We stick to Rosenberg—Ware's notations. In particular we denote by X the space of signatures of A and for a a unit of A by $W(a)$ the clopen set of all σ in X with $\sigma(a)=-1$.

Theorem 1. *If the sets $W(a)$ from a basis of $X(WAP)$, then every clopen set of X has the form $W(a)$ with some a in $U(A)$ (SAP).*

To prove this we look at the step (1)⇒(2) in Rosenberg—Ware's proof of their theorem 2.2. We arrive at the relation

$$2^m E \perp 2^m F \sim 2^{m+n}\langle 1\rangle$$

with the bilinear spaces E, F introduced there {$\sim$denotes Witt equivalence}. From this we obtain a Witt-equivalence

$$2^m(E\otimes G) \perp 2^m(F\otimes G) \sim 2^{m+n} G$$

of *quadratic* spaces (cf. e.g. [7, p. 111] for the tensor product of a bilinear and a quadratic space). Since the cancellation law holds true for quadratic spaces over A [2], this relation implies an isometry

$$2^m(E\otimes G) \perp 2^m(F\otimes G) \cong 2^{m+n} G \perp 2^{m+n} H,$$

with H the quadratic hyperbolic plane over A. By [1, Satz 2.7] there exists a unit c of A such that $-c$ is represented by $2^m(E\oplus G)$ and c is represented by $2^m(F\oplus G)$. Now using Lemma 1 and Lemma 2 we obtain as in Rosenberg—Ware's proof that

$$W(a_1)\cap \ldots \cap W(a_n) = W(c).$$

With more effort we obtain a proof of Theorem 1 more generally for A a connected semilocal ring equipped with an involution.

Let X denote the space of signatures of (A, J), cf. [6], and let X_0 denote the space of signatures of the fixed ring A_0 of J. Then X is a closed subspace of X_0, cf. [6, p. 211]. Finally we denote for c in A the norm $c\cdot J(c)$ by Nc.

Lemma 3. *Assume that all the fields $A_0/\mathfrak{m}$ with $\mathfrak{m}$ running through the finitely many maximal ideals of A_0 contain at least four elements. Let E be a bilinear space over A_0 such that the hermitian space $E\otimes_{A_0}A$ is Witt-equivalent to zero. Then there exists units $c_1, \ldots, c_r$ of A such that*

$$(*) \qquad \langle 1, Nc_1\rangle\otimes\ldots\otimes\langle 1, Nc_r\rangle\otimes E\sim 0$$

over A_0.

PROOF. The kernel of the natural map from $W(A_0)$ onto $W(A, J)$ is the ideal generated by the classes of the spaces $\langle 1, -Nc\rangle$ with c running through the units of A [6, Prop. 2.5]. Thus

$$E\sim\langle a_1\rangle\langle 1, -Nc_1\rangle\perp\ldots\perp\langle a_r\rangle\langle 1, -Nc_r\rangle$$

with units a_i of A_0 and units c_i of A. Clearly the relation $(*)$ above holds true with these units $c_1, \ldots, c_r$.

From the description of the kernel of the map from $W(A_0)$ onto $W(A, J)$ just given it is further clear, that under the assumptions about A_0 in Lemma 3 a signature σ of A_0 lies in X if and only if $\sigma(Nc)=1$ for every c in $U(A)$ [6, Cor. 2.6].

We now assume that the space X of signatures of (A, J) has the property WAP, saying that the clopen sets

$$W(a):=\{\sigma\in X|\sigma(a)=-1\}$$

with a in $U(A_0)$ form a basis of X. We want to prove that these sets $W(a)$ are already all clopen sets of X (SAP).

It suffices to find for given units $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of A_0 with

$$W(a_1)\cap\ldots\cap W(a_n)=W(b_1)\cup\ldots\cup W(b_n)$$

a unit c of A_0 such that this set coincides with $W(c)$ [8]. As in [8] we introduce the *bilinear* spaces

$$E:=\bigotimes_{i=1}^{n}\langle 1, -a_i\rangle,\quad F:=\bigotimes_{i=1}^{n}\langle 1, -b_i\rangle$$

over A_0, and as in [8] we see that

$$(2^m E\perp 2^m F)\otimes_{A_0}A\sim 2^{m+m}\langle 1\rangle$$

over (A, J) with m some natural number.

Assume now that all fields $A_0/\mathfrak{m}$ contain at least 4 elements. By Lemma 3 there exists a bilinear space

$$L:=\langle 1, Nc_1\rangle\otimes\ldots\otimes\langle 1, Nc_r\rangle$$

with units $c_1, \ldots, c_r$ of A such that

$$2^m E\otimes L\perp 2^m F\otimes L\sim 2^{m+n}L$$

over A_0. We multiply this relation with the binary quadratic space G from above and learn that the quadratic space

$$2^m E\otimes L\otimes G\perp 2^m F\otimes L\otimes G$$

is isotropic. Thus there exists a unit c of A_0 such that $2^m E\otimes L\otimes G$ represents $-c$ and $2^m F\otimes L\otimes G$ represents c. Now observing that $\sigma(L)=2^r$ and $\sigma(G)=2$ for any σ in X we see as in [8] that $W(a_1)\cap\ldots\cap W(a_n)$ coincides with $W(c)$. This finishes our proof that WAP implies SAP in the case that all fields $A_0/\mathfrak{m}$ contain at least 4 elements.

We shall obtain the proof in general immediately from this and the following lemma.

Lemma 4. *Let* (A, J) *be an arbitrary semilocal* [1]) *ring with involution. Let* C *denote the extension* $A[T]/(f(T))$ *with* [2]) $f(T)=T^3+6T^2+29T+1$, *and let* t *denote the image of* T *in* C. *Let* J' *denote the involution of* C *which extends* J *and maps* t *onto itself. Then every residue class field of the fixed ring*

$$C_0 = A_0[t] = A_0[T]/(f(T))$$

of J' *contains at least 7 elements. Every signature* σ *of* A_0 *extends to a unique signature of* C_0, *denoted by* σ', *and* σ *is a signature of* (A, J) *if and only if* σ' *is a signature of* (C, J'). *For the regular norm* $N_{C_0/A_0}(c)$ *of a unit* [3]) c *of* C_0 *we have*

$$\sigma'(c) = \sigma(N_{C_0/A_0}(c)).$$

The *proof of this lemma* will be given below. Let X' denote the space of signatures of (C, J') and for b a unit of C_0 let $W'(b)$ denote the set of all τ in X' with $\tau(b)=-1$. By our lemma we have a canonical homeomorphism $\sigma'\to\sigma$ from X' onto X and this homeomorphism maps $W'(b)$ to $W(N_{C_0/A_0}(b))$. Furthermore the inverse image of a subset $W(a)$ of X, a in $U(A_0)$, is the set $W'(a)$.

Thus it clearly suffices to prove $WAP\Rightarrow SAP$ for (C, J') instead for (A, J). This had been done above since all residue class fields of C_0 contain at least 4, in fact 7, elements.

It remains *to prove Lemma* 4. Our polynomial $f(T)=T^3+6T^2+29T+1$ is irreducible over the prime fields $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_5$. Thus indeed all residue class fields of A_0 contain at least seven elements. The other statements of Lemma 4 will now be proved by a transfer method similar to [4, § 3]. We introduce the A-linear form $s\colon C\to A$ with $s(1)=1$, $s(t)=s(t^2)=0$ and the restriction $s_0\colon C_0\to A_0$ of s to an A_0-linear form on C_0. The hermitian form $s(xJ'(y))$ on the A-module C is non degenerate and thus s induces a transfer map

$$s^*\colon W(C, J')\to W(A, J)$$

sending the class of an hermitian space (E, Φ) over (C, J') to the class of the hermitian space $(E, s\circ\Phi)$ over (A, J). In the same way we obtain a transfer map

$$s_0^*\colon W(C_0)\to W(A_0).$$

Since $[C_0\colon A_0]$ is odd every signature of A_0 extends to at least one signature of C_0 [6, p. 236]. The argument in [6] also shows, that every signature of A extends to at least one signature of C.

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[1]) The proof below shows that Lemma 4 remains true for A an arbitrary commutative ring with 1, if we use the more general concept of signatures developed in [4].

[2]) cf. the cubic polynomial in [3, p. 26].

[3]) $N_{C_0/A_0}(c)=$ determinant of the A_0-linear map $x\to cx$ on C_0.

We now consider a fixed signature σ of A_0 and a fixed extension τ of σ to C_0. We choose a homomorphism β from C_0 into a real closed field R inducing τ, i.e. such that the diagram

$$\begin{array}{ccc} W(C_0) & \xrightarrow{\beta_*} & W(R) \\ & \searrow^{\tau} \quad \swarrow^{\varrho}_{\cong} & \\ & \mathbf{Z} & \end{array}$$

commutes, with ϱ the unique signature of R, (cf. [4, § 4] for the existence of β). Let $\alpha\colon A \to R$ denote the restriction of β to A_0. Our polynomial $f(T)$ has precisely one root in the field $\mathbf{R}$ of real numbers {Observe that $f'(T)$ has no root in $\mathbf{R}$}. Thus $f(T)$ has also a unique root in R, and we learn that β is the only homomorphism from C_0 to R extending α.

There exist two more homomorphisms from C_0 to $R(\sqrt{-1})$ over α. Let γ be one of them. The commutative diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{(\beta,\gamma)} & R\times R(\sqrt{-1}) \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi'} \\ A_0 & \xrightarrow{\alpha} & R \end{array}$$

with φ the inclusion map and φ' the diagonal embedding is a tensor product diagram for C_0 and R over A_0, and we identify $R\times R(\sqrt{-1})$ with $C_0\otimes_{A_0}R$ in this way. The R-linear map $s_0\otimes\mathrm{id}$ from this tensor product to R decomposes into a pair of R-linear forms

$$s_1\colon R\to R, \quad s_2\colon R(\sqrt{-1})\to R,$$

and we have $s_1(x)=bx$ with some $b\neq 0$ in R. Since the diagram

$$\begin{array}{ccc} W(C_0) & \xrightarrow{(1\otimes\alpha)_*} & W(C_0\otimes R) \\ {\scriptstyle s_0^*}\uparrow & & \uparrow{\scriptstyle (s_0\otimes 1)^*} \\ W(A_0) & \xrightarrow[\alpha_*]{} & W(R) \end{array}$$

commutes we obtain for z in $W(C_0)$

$$\alpha_* s_0^*(z) = \langle b\rangle \beta_*(z) + s_2^*\gamma_*(z).$$

But the additive map s_2^* from $W(R(\sqrt{-1}))=\mathbf{Z}/2\mathbf{Z}$ to $W(R)=\mathbf{Z}$ must vanish, and we have

$$\alpha_* s_0^*(z) = \langle b\rangle \beta_*(z).$$

Applying ϱ we obtain, since $\varrho\circ\beta_*=\tau$, $\varrho\circ\alpha_*=\sigma$:

$$\sigma s_0^*(z) = \varrho(b)\,\tau(z).$$

Now the A_0-bilinear form $s_0(xy)$ on C_0 has with respect to the basis $1,\ -t,\ t^2$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

and thus s_0^* maps the unit element of $W(C_0)$ to the unit element of $W(A_0)$. Inserting $z=1$ in the formula above we see that $\varrho\,(b)=1$, hence

$$(*) \qquad \sigma s_0^*(z) = \tau(z)$$

for every z in $W(C_0)$. In particular τ must be the only signature of C_0 extending σ, since any other extension of σ has to fulfill the same equation (*). Now Lemma 4 is proved up to the last statement. For c a unit of C_0 we have

$$\alpha N_{C_0/A_0}(c) = \beta(c) N_{R(\sqrt{-1})/R}(\gamma(c)).$$

Applying ϱ we obtain indeed

$$\sigma N_{C_0/A_0}(c) = \tau(c).$$

Remark. It is now easy to deduce from (*) the analogous formula

$$\sigma s^*(w) = \tau(w)$$

for w in $W(C, J')$ and σ a signature of (C, J).

Apparently Lemma 4 is just a special case of a theory of signatures of "Frobenius extensions" which I hope to make explicit in the near future.

We now turn to Rosenberg—Ware's Theorem 3.1. The involution of A is assumed to be trivial, since we want to avoid the complications of "quadratic forms over a ring with involution".

Theorem 2. *The following statements are equivalent for A an arbitrary semilocal ring.*

SAP: Every clopen set of X has the form $W(a)$.

HMP: If a quadratic space E over A is indefinite at all signatures of A then mE is isotropic for some natural number m.

We first prove $SAP \Rightarrow HMP$. We need the following lemma which will be proved below.

Lemma 5. *If A is an arbitrary commutative ring and z is an element of the quadratic Witt group $WQ(A)$* [7, p. 112] *which has image zero in $W(A)$, then $8z=0$.*

Let E be a quadratic space of rank n over our semilocal ring A and assume $|\sigma(E)| \leqq n-2$ for every σ in X. Using SAP we produce as in [8, § 3] a bilinear space $F=\langle b_1, \ldots, b_{n-2}\rangle$ over A such that $\sigma(F)=\sigma(E)$ for all σ in X. By Lemma 2 the quadratic spaces $2E$ and $F\otimes G$, with G as above, have again the same value for all σ in X. Thus for some $m\geqq 0$ the bilinear spaces $2^{m+1}\tilde{E}$ and $2^m F\otimes\tilde{G}$ are Witt-equivalent. By Lemma 5 the quadratic spaces $2^{m+4}E$ and $2^{m+3}F\otimes G$ are Witt-equivalent. Since $2^{m+3}F\otimes G$ has smaller rank than $2^{m+4}E$ and cancellation holds true for quadratic spaces we see that $2^{m+4}E$ is isotropic.

The proof of Lemma 5 is easy. Let L denote the unique positive definite quadratic space of rank 8 over $\mathbf{Z}$. As is well known $\tilde{L} \sim 8\langle 1 \rangle$. Indeed, the natural map from $W(\mathbf{Z})$ to $W(\mathbf{R})$ is an isomorphism [7, p. 90]. If now E is a quadratic space over A with $\tilde{L} \sim 0$, then[4])

$$0 \sim \tilde{E} \otimes L \cong \tilde{L} \otimes E \sim 8E.$$

Remark. The cokernel of the canonical map from $WQ(A)$ to $W(A)$ is also killed by 8. Indeed for every bilinear space F over A we have

$$8F \sim (F \otimes L)^{\tilde{}}.$$

We finally prove $HMP \Rightarrow SAP$ along the lines of [8]. For arbitrary units a, b of A we have to find a unit c of A with $W(a) \cap W(b) = W(c)$. We introduce the quadratic space

$$E := \langle -1, a, b, ab \rangle \otimes G$$

and observe that $\sigma(E) = \pm 4$ for every σ in X. By (HMP) $2^m E$ is isotropic for some natural number m.

Assume now that all residue class fields of A have at least 3 elements. Then we obtain as in [8] by use of [1, Satz 2.7] an equation

$$t = s + bc$$

with units t, s, c of A such that $-t$ is represented by $2^m G$, s is represented by $2^m \langle a \rangle G$ and c is represented by $\langle 1, a \rangle G$. Using Lemma 1 we deduce from this equation as in [8] that $W(a) \cap W(b) = W(c)$.

If A has residue class fields with 2 elements, then we switch over to the cubic extension C of Lemma 4, the involution J there now being trivial. We have units t, s, c of C with $t = s + bc$ and $-t, s, c$ represented by the spaces listed above over C. Thus we have in the space X' of signatures of C

$$W'(a) \cap W'(b) = W'(c).$$

Applying the canonical homeomorphism from X' to X we obtain by Lemma 4

$$W(a) \cap W(b) = W(N_{C/A}(c)).$$

This finishes the proof of Theorem 2.

In Theorem 2 we considered a Hasse—Minkowski principle for *quadratic* spaces. Actually this Hasse—Minkowski principle is equivalent to the analogous principle for *bilinear* spaces. This follows immediately from Lemma 2 above and the following two observations:

Lemma 6. *Let E be a bilinear space over A such that the quadratic space $E \otimes G$ is isotropic. Then $6E$ is isotropic.*

This had been proved in [5, § 5].

[4]) We write $\tilde{E} \otimes L$ instead of $\tilde{E} \otimes (L \otimes_{\mathbf{Z}} A)$, etc.

Lemma 7. *Let E be a quadratic space over A such that the associated bilinear space $\tilde{E}$ is isotropic. Then $2E$ is isotropic.*

Indeed, E contains a primitive vector x with $2q(x)=0$. Thus the primitive vector $x\oplus x$ of the quadratic space $E\perp E$ is isotropic.

References

[1] R. BAEZA, M. KNEBUSCH, Annullatoren von Pfisterformen über semilokalen Ringen, *Math. Z.* **140** (1974), 41—62.
[2] M. KNEBUSCH, Isometrien über semilokalen Ringen, *Math. Z.* **108** (1969), 255—268.
[3] M. KNEBUSCH, Runde Formen über semilokalen Ringen, *Math. Ann.* **193** (1971), 21—34.
[4] M. KNEBUSCH, Real closures of commutative rings I, *J. reine angew. Math.* **274/275** (1975), 61—89.
[5] M. KNEBUSCH, Generalization of a theorem of Artin—Pfister to arbitrary semilocal rings, and related topics, *J. of Algebra* **36** (1975), 46—67.
[6] M. KNEBUSCH, A. ROSENBERG, R. WARE, Signatures on semilocal rings, *J. of Algebra* **26** (1973), 208—250.
[7] J. MILNOR, D. HUSEMOLLER, Symmetric bilinear forms, Ergebnisse Math. u. Grenzgeb. Bd 73, *Springer, New York—Heidelberg—Berlin* 1973.
[8] A. ROSENBERG, R. WARE, Equivalent topological properties of the space of signatures of a semilocal ring, *Publ. Math. (Debrecen)* **23** (1976), 283—289.

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