Non-additive ring and module theory I. General theory of monoids

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Much of the present theory of rings and modules depends only on the multiplicative structure, not on the additive structure of the objects in question. Another part of this theory depends mostly on the additive structure. This last part has extensively been studied in the context of abelian (or additive) categories. We want to introduce categorical tools to study just the multiplicative properties. A surprisingly large part of ring and module theory can be recovered in this way, e.g. Morita theorems, Brauer groups of Azumaya algebras, Maschke's theorem about separable group rings etc. Furthermore this generalized theory applies to a series of examples to be discussed later.

The general background of our theory is the notion of a monoidal category (not necessarily symmetric or closed or complete or cocomplete). This is a category $\mathscr{C}$ in which we can form associative "tensor products" of objects and morphisms, with "unit" $I \in \mathscr{C}$ such that $I \otimes X \cong X \cong X \otimes I$. Rings are generalized to monoids in $\mathscr{C}$, i.e. objects $A \in \mathscr{C}$ together with an associative unitary multiplication μ: $A \otimes A \to A$. Modules are generalized to A-objects, i.e. objects $M \in \mathscr{C}$ together with an associative unitary multiplication ν: $A \otimes M \to M$. We use a Yoneda Lemma like technique to do most computations elementwise. So the associativity of the monoid A, μ: $A \otimes A \to A$ can be expressed by $a(bc) = (ab)c$ for "elements" a, b, c of A (see § 1). Similarly an A-morphism f: $M \to N$ of A-objects M, N is described by $f(am) = af(m)$.

This concept may be applied to a series of examples some of which are rings and modules, coalgebras and comodules, Banach algebras and Banach modules, H-module algebras and modules over them for H a Hopf algebra, monoids (in sets) and monoid sets with equivariant maps etc.

In this paper we want to introduce the general background of non-additive ring and module theory, tensor products over arbitrary base-monoids (read: base-rings), and the technique of elementwise computation.

1. Notation and the universal property of the tensor product

Let $(\mathscr{C}, \otimes, I)$ be a *monoidal category*, i.e. a category $\mathscr{C}$ with a functor $\otimes : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ and an object $I\in\mathscr{C}$, and with natural isomorphisms

$$\alpha:\quad A\otimes(B\otimes C)\cong(A\otimes B)\otimes C$$

$$\lambda:\quad I\otimes A\cong A$$

$$\varrho:\quad A\otimes I\cong A$$

such that all diagrams

$$\begin{array}{ccccc}
A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha} & ((A\otimes B)\otimes C)\otimes D \\
\downarrow{\scriptstyle A\otimes\alpha} & & & & \uparrow{\scriptstyle \alpha\otimes D} \\
A\otimes((B\otimes C)\otimes D) & & \xrightarrow{\qquad\alpha\qquad} & & (A\otimes(B\otimes C))\otimes D
\end{array}$$

and

$$\begin{array}{ccc}
A\otimes(I\otimes B) & \xrightarrow{\alpha} & (A\otimes I)\otimes B \\
{\scriptstyle A\otimes\lambda}\searrow & & \swarrow{\scriptstyle \varrho\otimes B} \\
& A\otimes B &
\end{array}$$

and

$$\begin{array}{ccc}
I\otimes I & = & I\otimes I \\
{\scriptstyle \lambda}\searrow & & \swarrow{\scriptstyle \varrho} \\
& I &
\end{array}$$

commute.

If there is an additional natural isomorphism

$$\gamma:\ A\otimes B\cong B\otimes A$$

such that

$$\mathrm{id}_{A\otimes B}=(A\otimes B\xrightarrow{\gamma}B\otimes A\xrightarrow{\gamma}A\otimes B)$$

and

$$\begin{array}{ccc}
A\otimes I & \xrightarrow{\gamma} & I\otimes A \\
{\scriptstyle \varrho}\searrow & & \swarrow{\scriptstyle \lambda} \\
& A &
\end{array}$$

and

$$\begin{array}{ccccc}
A\otimes(B\otimes C) & \xrightarrow{\alpha} & (A\otimes B)\otimes C & \xrightarrow{\gamma} & C\otimes(A\otimes B) \\
\downarrow{\scriptstyle A\otimes\gamma} & & & & \downarrow{\scriptstyle \alpha} \\
A\otimes(C\otimes B) & \xrightarrow{\alpha} & (A\otimes C)\otimes B & \xrightarrow{\gamma\otimes B} & (C\otimes A)\otimes B
\end{array}$$

commute, then $\mathscr{C}$ is called a *symmetric monoidal category* [3].

Let $\mathscr{C}$ always be a monoidal category.

For objects $A, C \in \mathscr{C}$ we use the notation $A(X) := \mathscr{C}(X, A)$. So A will be considered as a contravariant functor from $\mathscr{C}$ to the category $\mathscr{S}$ of sets or as a "variable set". By the Yoneda Lemma [12] there is a bijection between the morphisms $f\colon A \to B$ in $\mathscr{C}$ and the natural transformations $\varphi\colon A(X) \to B(X)$ by $\varphi = \mathscr{C}(X, f)$, which then will be denoted by the same letter: $f\colon A(X) \to B(X)$. Observe then that $f(a) = f \circ a$ for $a \in A(X)$.

Consider $A \otimes B$ as a "variable set". We want to introduce a universal property for it, which resembles the usual universal property of the tensor product. For this purpose consider a "variable set" $A(X) \times B(Y)$ in the two "variables" X and in Y. A natural transformation $f\colon A(X) \times B(Y) \to \mathscr{F}(X \otimes Y)$, natural in X and Y, with a contravariant functor $\mathscr{F}\colon \mathscr{C} \to \mathscr{S}$ is called a *bimorphism*. We simply write it as $f\colon A \times B \to \mathscr{F}$.

Clearly

$$A(X) \times B(Y) = \mathscr{C}(X, A) \times \mathscr{C}(Y, B) \ni (a, b) \xrightarrow{\otimes} a \otimes b \in \mathscr{C}(X \otimes Y, A \otimes B) = A \otimes B(X \otimes Y)$$

is a bimorphism.

1.1 Lemma. *(Universal property for tensor products). For each bimorphism $f\colon A \times B \to \mathscr{F}$ there is a uniquely determined natural transformation $f^*\colon A \otimes B \to \mathscr{F}$, such that*

$$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & A \otimes B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f^*} \\ & & \mathscr{F} \end{array}$$

commutes.

Furthermore $A \times B \xrightarrow{\otimes} A \otimes B \xrightarrow{g} \mathscr{F}$ is a bimorphism for each natural transformation $g\colon A \otimes B \to \mathscr{F}$.

Proof. Let $X = A$, $Y = B$ and $f' = f(\mathrm{id}_A, \mathrm{id}_B)$. Then $f' \in \mathscr{F}(A \otimes B)$ may be considered as a natural transformation $f^*\colon A \otimes B(Z) \to \mathscr{F}(Z)$ by the Yoneda Lemma namely $f^* = \mathscr{F}(-)(f')$. Now let $X, Y \in \mathscr{C}$ be arbitrary objects. For $(a, b) \in A(X) \times \times B(Y)$ we have

$$\begin{array}{ccc} A(A) \times B(B) & \xrightarrow{f} & \mathscr{F}(A \otimes B) \\ \downarrow{\scriptstyle A(a) \times B(b)} & & \downarrow{\scriptstyle \mathscr{F}(a \otimes b)} \\ A(X) \times B(Y) & \xrightarrow{f} & \mathscr{F}(X \otimes Y) \end{array}$$

commutative, hence for $(\mathrm{id}_A, \mathrm{id}_B) \in A(A) \times B(B)$ we get

$$\mathscr{F}(a \otimes b) \circ f(\mathrm{id}_A, \mathrm{id}_B) = f \circ A(a) \times B(b)(\mathrm{id}_A, \mathrm{id}_B) = f(a, b)$$

hence $(f^* \circ \otimes)(a, b) = f^*(a \otimes b) = \mathscr{F}(a \otimes b)(f') = \mathscr{F}(a \otimes b) \circ f(\mathrm{id}_A, \mathrm{id}_B) = f(a, b)$. Thus the diagram of the Lemma commutes.

Let g be such that $g \circ \otimes = f$. Then $g(\mathrm{id}_A \otimes \mathrm{id}_B) = (g \circ \otimes)(\mathrm{id}_A, \mathrm{id}_B) = f(\mathrm{id}_A, \mathrm{id}_B) = f'$ in $\mathscr{F}(A \otimes B)$, hence the induced natural transformations f^* and g from $A \otimes B$ to $\mathscr{F}$ are equal.

So we have a bijection between bimorphisms $f\colon A \times B \to \mathscr{F}$ and natural transformations $f^*\colon A \otimes B \to \mathscr{F}$. In particular for each object $C \in \mathscr{C}$ and bimorphism $f\colon A \times B \to C$ there is a unique morphism $f^*\colon A \otimes B \to C$ such that

commutes for all $X, Y \in \mathscr{C}$.

Certain, but usually not all, elements in $A \otimes B(X \otimes Y)$ are of the form $a \otimes b$ with $a \in A(X)$, $b \in B(Y)$. For those elements we have $f(a, b) = f^*(a \otimes b)$. Since f^* is uniquely determined by f, we want to describe $f^*(z)$ for all $z \in A \otimes B(X)$ with the use of f. Since f has two variables we introduce the notation

$$f(z_A, z_B) := f^*(z).$$

If we take in particular $\otimes\colon A \times B \to A \otimes B$ as bimorphism, then it clearly factors through $\mathrm{id}\colon A \otimes B \to A \otimes B$. With the notation just introduced we get $z = z_A \otimes z_B$ for all $z \in A \otimes B(X)$. The elements z which occur as tensor products $a \otimes b$ will be called *decomposable tensors*, the others *indecomposable*.

Since for any bimorphism $f\colon A \times B \to C$ the value $f(z_A, z_B)$ is uniquely determined for $z = z_A \otimes z_B$, if one only knows $f(a, b)$ for all $(a, b) \in A(X) \times B(Y)$, we will not make any difference between decomposable tensors $a \otimes b$ and indecomposable tensors $z_A \otimes z_B$ and thus will use the same notation $a \otimes b$ in both cases.

We will list a few interesting examples of monoidal categories, to which we will apply the later results:

a) Category of K-modules for a commutative ring K with the usual tensor product and $K \in K$-Mod.
b) The dual of K-Mod with the tensor product and $K \in K$-Mod.
c) Any category $\mathscr{C}$ which has finite products or coproducts and a final or initial object, in particular the categories of sets, R-modules or the category of small categories.
d) The category of endofunctors of a category $\mathscr{C}$ with composition and the identity functor.
e) The category of $R-R$-bimodules with the tensor product $M \otimes_R N$ and $R \in R$-Mod, where R is a possibly non-commutative ring.
f) The category of real or complex Banach spaces together with the completed tensor product and **R** resp. **C**. The morphisms in this category shall be the linear maps of norm less than or equal to 1.

Observe that the examples a), b), c) and f) are symmetric monoidal categories, whereas d) and e) are in general nonsymmetric.

2. Monoids and objects over monoids

A *monoid* in $\mathscr{C}$ is an object $A \in \mathscr{C}$ together with two morphisms $\mu\colon A\otimes A \to A$ and $\eta\colon I \to A$, such that the diagrams

$$\begin{array}{ccccc} A\otimes(A\otimes A) & \xrightarrow{\alpha} & (A\otimes A)\otimes A & \xrightarrow{\mu\otimes A} & A\otimes A \\ \big\downarrow{\scriptstyle A\otimes\mu} & & & & \big\downarrow{\scriptstyle \mu} \\ A\otimes A & & \xrightarrow{\qquad\quad \mu \qquad\quad} & & A \end{array}$$

and

$$(*)\qquad \begin{array}{ccccc} I\otimes A & \xrightarrow{\eta\otimes A} & A\otimes A & \xleftarrow{A\otimes\eta} & A\otimes I \\ & {\scriptstyle\lambda}\searrow & \big\downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\varrho} & \\ & & A & & \end{array}$$

commute.

If we write $\mu(a\otimes b)=ab$ for $(a, b)\in A(X)\times A(Y)$, then the commutativity of the first diagram is equivalent to $a(bc)=A(\alpha)\big((ab)c\big)\in A\big(X\otimes(Y\otimes Z)\big)$ for all $X, Y, Z\in\mathscr{C}$ and all $(a, b, c)\in A(X)\times A(Y)\times A(Z)$. As long as it is clear from the context we'll omit $A(\alpha)$ and simply write $\big((ab)c\big)\in A\big(X\otimes(Y\otimes Z)\big)$, hence $(ab)c=a(bc)$.

In the sequel we shall often omit composites of α, λ, ϱ and γ on the side of the domain of a composition of morphisms. This can be done without any harm by the coherence results of [5]. So we'll identify certain morphisms with different domains, or rather pick a fixed domain out of the set of possible uniquely isomorphic domains, using the axiom of choice. Hence when we talk about $A(X\otimes Y\otimes Z)$, then $X\otimes Y\otimes Z$ is such a fixed domain.

Since

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ {\scriptstyle\lambda^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle\lambda} \\ I\otimes X & \xrightarrow{I\otimes a} & I\otimes A \end{array}$$

commutes, we get $A(\lambda^{-1})\circ\lambda(\mathrm{id}_I\otimes a)=a$. Hence the commutativity of the left triangle of $(*)$ is equivalent to $A(\lambda^{-1})\big(\eta(\mathrm{id}_I)a\big)=a$ for all $X\in\mathscr{C}$ and all $a\in A(X)$. Again we will omit $A(\lambda^{-1})$ and obtain $\eta(\mathrm{id}_I)a=a$. If we write $1=\eta(\mathrm{id}_I)$, we get $1\cdot a=a$ for all $a\in A(X)$. The right triangle of $(*)$ is commutative iff $a\cdot 1=a$ for all $a\in A(X)$ (again omitting $A(\varrho^{-1})$).

Observe that for an arbitrary monoidal category $(\mathscr{C}, \otimes, I)$ there is no sense in asking if a monoid (A, μ, η) is commutative, since there is no symmetry $\gamma\colon A\otimes A\cong$ $\cong A\otimes A$. If, however, $\mathscr{C}$ is a symmetric monoidal category then a monoid (A, μ, η) is called *commutative* if

$$\begin{array}{ccc} A\otimes A & & \\ \big\downarrow{\scriptstyle\gamma} & \searrow{\scriptstyle\mu} & \\ & & A \\ A\otimes A & \nearrow{\scriptstyle\mu} & \end{array}$$

commutes. This is equivalent to $ab=ba$ for all $a\in A(X)$, $b\in A(Y)$ (again omitting $A(\gamma)$: $A(X\otimes Y)\to A(Y\otimes X)$).

Now let A be a monoid. An *A-left-object* in $\mathscr{C}$ is an object $M\in\mathscr{C}$ together with a morphism ν: $A\otimes M\to M$ such that

$$\begin{array}{ccccc} A\otimes(A\otimes M) & \xrightarrow{\alpha} & (A\otimes A)\otimes M & \xrightarrow{\mu\otimes M} & A\otimes M \\ \downarrow{\scriptstyle A\otimes\nu} & & & & \downarrow{\scriptstyle \nu} \\ A\otimes M & & \xrightarrow{\qquad\nu\qquad} & & M \end{array}$$

and

$$\begin{array}{ccc} I\otimes M & \xrightarrow{\eta\otimes M} & A\otimes M \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & M & \end{array}$$

commute. Writing $\nu(a\otimes m)=am$, this is equivalent to $(ab)m=a(bm)$ and $1\cdot m=m$ for all $m\in M(Z)$, $a\in A(X)$, $b\in A(Y)$. Here again we omit $M(\alpha)$ and $M(\lambda^{-1})$ or $M(\lambda)$.

A *monoid-morphism* f: $A\to B$ is a morphism $f\in\mathscr{C}$, such that

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & B\otimes B \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow{f} & B \end{array}$$

and

$$\begin{array}{ccc} & I & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle\eta} \\ A & \xrightarrow{f} & B \end{array}$$

commute, equivalently

$$f(ab)=f(a)f(b) \quad \text{and} \quad f(1)=1$$

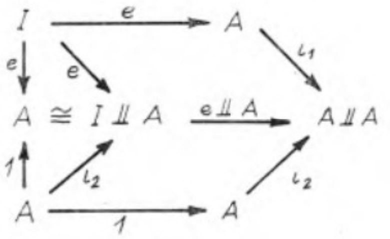
for all $a\in A(X)$, $b\in A(Y)$ and all $X, Y\in\mathscr{C}$.

An *A-morphism* f: $M\to N$ is a morphism $f\in\mathscr{C}$, such that

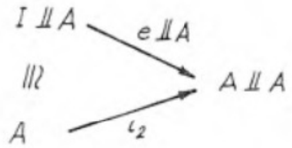
$$\begin{array}{ccc} A\otimes M & \xrightarrow{A\otimes f} & A\otimes N \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{f} & N \end{array}$$

commutes, equivalently $f(am)=af(m)$ for all $a\in A(X)$, $m\in M(Y)$ and all $X, Y\in\mathscr{C}$. Thus we get a category ${}_A\mathscr{C}$ of A-objects and A-morphisms for each monoid A and a category Mon $(\mathscr{C})$ of monoids and monoid-morphisms.

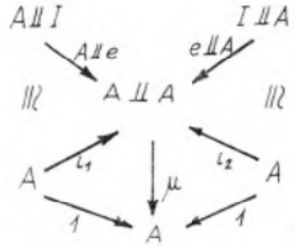
If the tensor product of $\mathscr{C}$ is the coproduct and I the initial object of $\mathscr{C}$, then the unique morphism $I \to X$ will be denoted by e. The diagrams



hence



commute. So the diagram



determines a unique $\mu\colon A \amalg A \to A$ which is a multiplication with 2-sided unit. It is easy to see that μ, the codiagonal, is associative. So each object in $\mathscr{C}$ carries a unique structure of a monoid. Since the codiagonal is a natural transformation each morphism is a monoid morphism. Thus $\operatorname{Mon}(\mathscr{C}) \cong \mathscr{C}$.

On the other hand let $(\mathscr{C}, \otimes, I)$ a symmetric monoidal category then $\operatorname{Mon}(\mathscr{C})$ is a symmetric monoidal category with the tensor product of $A, B \in \operatorname{Mon}(\mathscr{C})$ the tensor product in $\mathscr{C}$. In fact if $A, B \in \operatorname{Mon}(\mathscr{C})$, then $\mu_{A\otimes B}\colon (A\otimes B)\otimes(A\otimes B) \cong$ $\cong (A\otimes A)\otimes(B\otimes B) \xrightarrow{\mu_A\otimes\mu_B} A\otimes B$ where the isomorphism is the unique isomorphism defined by the coherence theorem of symmetric monoidal categories: $\varphi\colon (A\otimes B)\otimes$ $\otimes(A'\otimes B') \cong (A\otimes A')\otimes(B\otimes B')$. So $\mu_{A\otimes B}(a\otimes b\otimes a'\otimes b') = aa'\otimes bb'$. Now it's easy to check that $A\otimes B$ is a monoid. Furthermore $\gamma\colon A\otimes B \cong B\otimes A$ is a monoid morphism hence $\operatorname{Mon}(\mathscr{C})$ is a symmetric monoidal category.

If we form $\operatorname{Mon}(\operatorname{Mon}(\mathscr{C}))$ then this is again a symmetric monoidal category. Let $(A, \mu_A\colon A\otimes A \to A)$ be in $\operatorname{Mon}(\operatorname{Mon}(\mathscr{C}))$ and let $\bar{\mu}_A\colon A\otimes A \to A$ be the multiplication of A in $\operatorname{Mon}(\mathscr{C})$ then μ_A in particular is a monoid morphism, i.e.

$$\begin{array}{ccccc}
(A\otimes A)\otimes(A\otimes A) & \underset{\varphi}{\cong} & (A\otimes A)\otimes(A\otimes A) & \xrightarrow{\bar{\mu}_A\otimes\bar{\mu}_A} & A\otimes A \\
\downarrow{\scriptstyle \mu_A\otimes\mu_A} & & & & \downarrow{\scriptstyle \mu_A} \\
A\otimes A & & \xrightarrow{\quad\bar{\mu}_A\quad} & & A
\end{array}$$

commutes or if $\mu_A(a\otimes b)=: a\circ b$ and $\bar{\mu}_A(a\otimes b)=: ab$, then $(aa')\circ(bb')=(a\circ b)(a'\circ b')$. Observe now that $\eta: I\to A$ as morphism in $\text{Mon}(\mathscr{C})$ is a monoid morphism. Let $\bar{\eta}: I\to A$ be the unit of A in $\text{Mon}(\mathscr{C})$. Hence

$$\begin{array}{ccc} I & = & I \\ & \bar{\eta}\searrow \quad \swarrow\eta & \\ & A & \end{array}$$

commutes or $\eta=\bar{\eta}$. So the units for A in $\text{Mon}(\mathscr{C})$ and for A in $\text{Mon}(\text{Mon}(\mathscr{C}))$ are the same. Let $1\in A(I)$ be the unit and $a'=1=b$, then

$$a\circ b' = (a1)\circ(1b') = (a\circ 1)(1\circ b') = ab', \quad \text{hence} \quad \mu_A = \bar{\mu}_A.$$

Furthermore $a'b=(1a')(b1)=(1b)(a'1)=ba'$, hence the multiplication $\bar{\mu}_A$ was commutative to start with. Conversely if $\bar{\mu}_A$ is commutative then $\bar{\mu}_A$ is a monoid morphism, i.e. the commutative monoids are precisely the monoids in $\text{Mon}(\mathscr{C})$. Hence $\text{Mon}(\text{Mon}(\mathscr{C}))\cong C\,\text{Mon}(\mathscr{C})$, where $C\,\text{Mon}(\mathscr{C})$ is the full subcategory of $\text{Mon}(\mathscr{C})$ of commutative monoids.

Now it's easy to see that the tensor product in $C\,\text{Mon}(\mathscr{C})$ is a coproduct for $C\,\text{Mon}(\mathscr{C})$. The injections are

$$A \cong A\otimes I \xrightarrow{A\otimes I} A\otimes B.$$

Let $f: A\to C$ and $g: B\to C$ be morphisms in $C\,\text{Mon}(\mathscr{C})$. Then $h: A\otimes B\to C$ defined by $h(a\otimes b)=f(a)g(b)$ is a morphism in $C\,\text{Mon}(\mathscr{C})$ since $h(aa'\otimes bb')= =f(aa')\,g(bb')=f(a)\,f(a')\,g(b)\,g(b')=f(a)\,g(b)\,f(a')\,g(b')=h(a\otimes b)\,h(a'\otimes b')$ and $h(1\otimes 1)=f(1)g(1)=1$. If $h': A\otimes B\to C$ reduces to f on A and g on B then $h'(a\otimes 1)=f(a)$, $h'(1\otimes b)=g(b)$ and $h'(a\otimes b)=h'(a\otimes 1)(1\otimes b)=f(a)g(b)=h(a\otimes b)$ hence $h=h'$. We have proved

2.1: Theorem. 1) *Let $\mathscr{C}$ be a symmetric monoidal category then* $\text{Mon}(\text{Mon}(\mathscr{C}))= =C\,\text{Mon}(\mathscr{C})$ *has the tensor product of $\mathscr{C}$ as product.*

2) *If $\mathscr{C}$ is a monoidal category which has the coproduct as tensor product then $C\,\text{Mon}(\mathscr{C})\cong\text{Mon}(\mathscr{C})\cong\mathscr{C}$, i.e. each object of $\mathscr{C}$ carries a unique monoid structure, which is commutative.*

A special case of this is the fact the usual tensor product is a coproduct in the category of commutative K-algebras, also the usual tensor product is a product in the category of cocommutative K-coalgebras.

This uses the fact that the (commutative) monoids in K-Mod resp. $K\text{-Mod}^{\text{op}}$ (examples a) and b)) are the (commutative) K-algebras resp. the (cocommutative) K-coalgebras. ${}_A\mathscr{C}$ is then the category of A-modules for the K-algebra A resp. the category of K-comodules for the K-coalgebra A. For the last assertion observe that for $M, N\in {}_AK\text{-Mod}^{\text{op}}={}_A\mathscr{C}$ we have $f\in {}_A\mathscr{C}(M, N)$ iff

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ A\otimes N & \rightarrow & A\otimes M \end{array}$$

commutes in K-Mod.

For the example c) in the special case (Sets, $\times$, $\{\emptyset\}$) the monoids are precisely the (abstract) monoids and ${}_A\mathscr{C}$ is the category of sets on which the monoid A acts, together with the equivariant maps. For $(\mathscr{C}, \times, E)$ with E a final object one obtains the categorical monoids.

In example d) the monoids are precisely the monads $\mathscr{H}$. An example for a left-$\mathscr{H}$-object, if $\mathscr{H}$ comes from a pair of adjoint functors $\mathscr{F}: \mathscr{C} \to \mathscr{C}$ and $\mathscr{G}: \mathscr{C} \to \mathscr{C}$ [12], is the functor $\mathscr{G}$. The functor $\mathscr{F}$ is a right $\mathscr{H}$-object, and $\mathscr{H} = \mathscr{G} \otimes \mathscr{F}$ as two-sided $\mathscr{H}$-objects.

The monoids for example e) are the R-rings in the sense of P. M. COHN [see also SWEEDLER 15].

For example f) the monoids are precisely the Banach algebras.

2.2. Theorem. *The forgetful functor* $\mathscr{U}: {}_A\mathscr{C} \to \mathscr{C}$ *is monadic.*

PROOF. Since A is a monoid, the functor $A \otimes -: \mathscr{C} \to \mathscr{C}$ is a monad with $\mu: (A\otimes)(A\otimes) \to A\otimes$ and $\eta: I\otimes \to A\otimes$ induced by $\mu: A \otimes A \to A$ and $\eta: I \to A$. Furthermore $I\otimes$ is identified by λ with $\mathrm{Id}_{\mathscr{C}}$.

The "algebras" for the monad $A\otimes$ are [12] pairs (M, ν) where $M \in \mathscr{C}$ and $\nu: A \otimes M \to M$ is a morphism in $\mathscr{C}$ such that

$$\begin{array}{ccc} A\otimes(A\otimes M) & \xrightarrow{A\otimes\nu} & A\otimes M \\ {\scriptstyle \mu(M)}\downarrow & & \downarrow{\scriptstyle \nu} \\ A\otimes M & \xrightarrow{\nu} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{\eta(M)} & A\otimes M \\ {\scriptstyle id_M}\downarrow & \swarrow{\scriptstyle \nu} & \\ M & & \end{array}$$

commute. This clearly is equivalent to the definition of an A-object. An "algebra morphism" for $A\otimes$ is a morphism $f: M \to N$, such that

$$\begin{array}{ccc} A\otimes M & \xrightarrow{A\otimes f} & A\otimes N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commutes. Again this is equivalent to $f: M \to N$ being an A-morphism. Hence ${}_A\mathscr{C} \to \mathscr{C}$ is monadic with left-adjoint $\mathscr{C} \ni M \to A \otimes M \in {}_A\mathscr{C}$.

2.3. Corollary: *I is a monoid in $\mathscr{C}$ and the category ${}_I\mathscr{C}$ is equivalent to $\mathscr{C}$.*

PROOF. Let $\mu: I \otimes I \to I$ be λ or ϱ and $\eta: I \to I$ be the identity. Then by the coherence theorem [5] I is a monoid. Since $I\otimes \cong \mathrm{Id}_{\mathscr{C}}$ by λ, both have equivalent categories of algebras, ${}_I\mathscr{C}$ for the functor $I\otimes$ and $\mathscr{C}$ for the functor $\mathrm{Id}_{\mathscr{C}}$.

A special instance of this is the fact that the category of abelian groups "is" the category of $\mathbf{Z}$-modules, since $(\mathrm{Ab}, \otimes, \mathbf{Z})$ is a monoidal category.

2.4. Corollary: *The forgetful functor $\mathscr{U}: {}_A\mathscr{C} \to \mathscr{C}$ creates limits and isomorphisms. If $\mathscr{C}$ is complete so is ${}_A\mathscr{C}$ and the limits are formed in $\mathscr{C}$. $\mathscr{U}$ creates those colimits which are preserved by $A\otimes$. If $\mathscr{C}$ has a generator then ${}_A\mathscr{C}$ also has a generator.*

Although we will not use the following facts they should be stated. The straight forward proofs are left to the reader.

2.5. Proposition: *The forgetful functor* $\mathscr{U}$: Mon $(\mathscr{C}) \to \mathscr{C}$ *creates limits and isomorphisms. It even creates difference cokernels of* $\mathscr{U}$*-contractible pairs.*

The functor $\mathscr{U}$: Mon $(\mathscr{C}) \to \mathscr{C}$, however, has in general no left adjoint, e.g. the category of finite sets with the product as $\otimes$.

3. Inner hom-functors and tensor products

Let A, B be monoids in the monoidal category $\mathscr{C}$. An $A-B$*-biobject* is an object $M \in \mathscr{C}$ together with morphisms $A \otimes M \to M$ and $M \otimes B \to M$ such that M becomes an A-left-object, a B-right-object and the diagram

$$\begin{array}{ccc} A \otimes M \otimes B & \to & M \otimes B \\ \downarrow & & \downarrow \\ A \otimes M & \longrightarrow & M \end{array}$$

commutes, i.e. $a(mb) = (am)b$. A morphism of $A-B$-biobjects is a morphism in $\mathscr{C}$, which is an A-left and B-right morphism, i.e. $f(am) = af(m)$ and $f(mb) = = f(m)b$. The category of $A-B$-biobjects is denoted by ${}_A\mathscr{C}_B$. Clearly there are forgetful functors $\mathscr{U}$: ${}_A\mathscr{C}_B \to {}_A\mathscr{C}$ resp. $\mathscr{U}$: ${}_A\mathscr{C}_B \to \mathscr{C}_B$. These functors are monadic in the same way as the functor in Theorem 2.2. Observe that $\mathscr{U}$ does not in general preserve colimits.

Assume that $\mathscr{C}$ and ${}_B\mathscr{C}$ have difference cokernels. Let $M \in {}_B\mathscr{C}_A$ and $N \in {}_A\mathscr{C}$, then we consider the two morphisms in ${}_B\mathscr{C}$

$$f: M \otimes (A \otimes N) \xrightarrow{M \otimes \nu_N} M \otimes N$$

and

$$g: M \otimes (A \otimes N) \xrightarrow{\alpha} (M \otimes A) \otimes N \xrightarrow{\nu_M \otimes N} M \otimes N.$$

The difference cokernel of f and g in ${}_B\mathscr{C}$ will be denoted by

$$M \otimes (A \otimes N) \underset{g}{\overset{f}{\rightrightarrows}} M \otimes N \xrightarrow{{}_B\omega} M \otimes_A N.$$

If we consider f and g as morphisms in $\mathscr{C}$, we get a difference cokernel $\omega: M \otimes N \to \to M \otimes_A N$ through which ${}_B\omega$ factors uniquely:

$$\begin{array}{ccccc} M \otimes (A \otimes N) & \underset{g}{\overset{f}{\rightrightarrows}} & M \otimes N & \xrightarrow{\omega} & M \otimes_A N \\ & & & {}_B\omega \searrow & \downarrow \\ & & & & M \otimes_A N \end{array}$$

Observe that we use the same notation for the objects of the difference cokernels in $\mathscr{C}$ and in ${}_B\mathscr{C}$ though $\mathscr{U}(M \otimes_A N) \ncong M \otimes_A N$ in general for $\mathscr{U}$: ${}_B\mathscr{C} \to \mathscr{C}$ the forgetful functor. If, however, $B \otimes : \mathscr{C} \to \mathscr{C}$ preserves difference cokernels then $\mathscr{U}(M \otimes_A N) \cong \cong M \otimes_A N$. This is the case if tensor products in $\mathscr{C}$ preserve difference cokernels, so for example in symmetric closed monoidal categories which are discussed later on in this paragraph. If in addition $N \in {}_A\mathscr{C}_C$ then ${}_B\omega_C$: $M \otimes N \to M \otimes_A N$ denotes the difference cokernel of f and g in ${}_B\mathscr{C}_C$.

Let $M \in {}_B\mathscr{C}_A$, $N \in {}_A\mathscr{C}$ and $P \in {}_B\mathscr{C}$. A bimorphism $f\colon M \times N \to P$ is called a *B-left A-bimorphism* if $f(ma, n) = f(m, an)$ and $f(bm, n) = bf(m, n)$ for all $m \in M(X)$, $n \in N(Y)$, $a \in A(Z)$, $b \in B(U)$.

3.1. Proposition: *Let $M \in {}_B\mathscr{C}_A$, $N \in {}_A\mathscr{C}$ and $P \in {}_B\mathscr{C}$. For each B-left A-bimorphism $f\colon M \times N \to P$ there is a unique morphism $g\colon M \otimes_A N \to P$ in ${}_B\mathscr{C}$ such that the diagrams*

$$\begin{array}{ccccc} M(X) \times N(Y) & \xrightarrow{\otimes} & M \otimes N(X \otimes Y) & \xrightarrow{{}_B\omega} & M \otimes_A N(X \otimes Y) \\ & \searrow^{f} & & & \downarrow g \\ & & & & P(X \otimes Y) \end{array}$$

commute.

Proof. By Lemma 1.1 f may be factored uniquely through $\otimes$ by a morphism $f^*\colon M \otimes N \to P$. f^* is in ${}_B\mathscr{C}$ because of $bf(m, n) = f(bm, n)$. Now $f(ma, n) = f(m, an)$ implies $f^*(ma \otimes n) = f^*(m \otimes an)$ hence $f^* \circ (\nu_M \otimes N) \circ \alpha = f^* \circ (M \otimes \nu_N)$. So f^* can be uniquely factored through ${}_B\omega$.

Let $(\mathscr{C}, \otimes, I)$ be a monoidal category. If the functor $X\otimes\colon \mathscr{C} \ni Y \to X \otimes Y \in \mathscr{C}$ has a right adjoint it will be denoted by $[X, -]$, so that

$$\mathscr{C}(X \otimes Y, Z) \cong \mathscr{C}(Y, [X, Z])$$

natural in Y and Z. If $X\otimes$ has a left adjoint it will be denoted by $\langle X, -\rangle$, so that

$$\mathscr{C}(Y, X \otimes Z) \cong \mathscr{C}(\langle X, Y\rangle, Z)$$

natural in Y and Z. If $(\mathscr{C}, \otimes, I)$ has right adjoint functors to $X\otimes$ for all $X \in \mathscr{C}$, then $\mathscr{C}$ is called a *(left-) closed monoidal category*. If $(\mathscr{C}, \otimes, I)$ has left adjoint functors to $X\otimes$ for all $X \in \mathscr{C}$, then $\mathscr{C}$ is dual to a closed monoidal category, which we shall call a *(left-) coclosed monoidal category*.

The most interesting example for this is the dual of the category of modules over a commutative ring, which is used for studying coalgebras.

If $\otimes X$ has a right adjoint for all $X \in \mathscr{C}$, then $\mathscr{C}$ is called a *(right-) closed monoidal category*. If $\mathscr{C}$ is symmetric, then $\mathscr{C}$ is left-closed iff $\mathscr{C}$ is right-closed. We will only consider monoidal categories which are left-closed and call them closed monoidal categories.

Let $M, N \in {}_A\mathscr{C}$, $X \in \mathscr{C}$. Then $M \otimes X$ is again in ${}_A\mathscr{C}$. So $M\otimes\colon \mathscr{C} \to {}_A\mathscr{C}$ is a functor. We want to find a right adjoint to it, which will be denoted by ${}_A[M, N]$ hence

$${}_A\mathscr{C}(M \otimes X, N) \cong \mathscr{C}(X, {}_A[M, N]) = {}_A[M, N](X).$$

We observe that there are two morphisms $u, v\colon \mathscr{C}(M, N) \to \mathscr{C}(A \otimes M, N)$, namely $f \mapsto f \circ \nu_M$ and $f \mapsto \nu_N \circ (A \otimes f)$. By definition f is in ${}_A\mathscr{C}(M, N)$ iff these two maps coincide on f iff

$$\begin{array}{ccc} A \otimes M & \xrightarrow{A \otimes f} & A \otimes N \\ \downarrow \nu_M & & \downarrow \nu_N \\ M & \xrightarrow{f} & N \end{array}$$

commutes. Hence we get ${}_A\mathscr{C}(M, N)$ as a difference kernel:

$$ {}_A\mathscr{C}(M, N) \xrightarrow{\omega} \mathscr{C}(M, N) \overset{u}{\underset{v}{\rightrightarrows}} \mathscr{C}(A\otimes M, N). $$

Let $\mathscr{C}$ be a closed monoidal category with difference kernels. Let u', v': $[M, N](X)\cong \cong\mathscr{C}(M\otimes X, N) \overset{u}{\underset{v}{\rightrightarrows}} \mathscr{C}((A\otimes M)\otimes X, N)\cong[A\otimes M, N](X)$. Let $K(M, N)$ be the difference kernel of u', v': $[M, N]\to[A\otimes M, N]$. Then we get a commutative diagram of difference kernels

$$ \begin{array}{ccccc} {}_A\mathscr{C}(M\otimes X, N) & \longrightarrow & \mathscr{C}(M\otimes X, N) & \overset{u}{\underset{v}{\rightrightarrows}} & \mathscr{C}((A\otimes M)\otimes X, N) \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ \mathscr{C}(X, K(M,N)) & \longrightarrow & \mathscr{C}(X, [M,N]) & \overset{u'}{\underset{v'}{\rightrightarrows}} & \mathscr{C}(X, [A\otimes M, N]). \end{array} $$

Hence $M\otimes$: $\mathscr{C}\to{}_A\mathscr{C}$ has a right adjoint, so $K(M, N)\cong{}_A[M, N]$, i.e.

$$ {}_A[M, N]\longrightarrow[M, N] \overset{u'}{\underset{v'}{\rightrightarrows}} [A\otimes M, N] $$

is a difference kernel.

Here we use the fact that $\mathscr{C}$ is left closed. In fact if the operation of A on M is right sided, then we cannot form $\mathscr{C}_A(M\otimes X, N)$. For this we need commutativity of

$$ \begin{array}{ccc} M\otimes A\otimes X & \to & N\otimes A\otimes X \\ \downarrow & & \downarrow \\ M\otimes X & \longrightarrow & N\otimes X \end{array} \quad \text{or of} \quad \begin{array}{ccc} M\otimes X\otimes A & \to & N\otimes X\otimes A \\ \downarrow & & \downarrow \\ M\otimes X & \longrightarrow & N\otimes X \end{array} $$

but in the first diagram the upper horizontal morphism and in the second diagram the two vertical morphisms cannot be defined in general.

Dual to the above construction we get a difference cokernel

$$ \langle A\otimes M, N\rangle \rightrightarrows \langle M, N\rangle \longrightarrow {}_A\langle N, M\rangle, $$

although we cannot immediately dualize the above construction, since A then becomes a comonoid. But the only thing we have used is that there is a morphism $A\otimes M\to M$, not that A is a monoid.

If $M\otimes$: $\mathscr{C}\to{}_A\mathscr{C}$ has a right adjoint, then we get the adjunction morphisms $X\to{}_A[M, M\otimes X]$ and $M\otimes{}_A[M, N]\to N$ the last morphism in ${}_A\mathscr{C}$. It induces a bimorphism $M\times{}_A[M, N]\to N$ which we write as $(m, f)\to\langle m\rangle f$. Since the morphism $M\otimes{}_A[M, N]\to N$ is an A-morphism we get $\langle am\rangle f=a(\langle m\rangle f)$ for $a\in A(X)$, $m\in M(Y)$ and $f\in{}_A[M, N](Z)$. Conversely if $f\in[M, N](Z)$ and $\langle am\rangle f=a(\langle m\rangle f)$ for all $m\in M(Y)$, $a\in A(X)$ then $f\in{}_A[M, N](Z)$ by the difference kernel property.

Since the diagram

$$ \begin{array}{ccc} M(X)\times{}_A[M,N](Y) & \xrightarrow{\psi} & N(X\otimes Y) \\ \wr\Vert & & \Vert \\ \mathscr{C}(X, M)\times{}_A\mathscr{C}(M\otimes Y, N) & \xrightarrow{\varphi} & \mathscr{C}(X\otimes Y, N) \end{array} $$

with $\varphi(m, f)=(X\otimes Y \xrightarrow{m\otimes Y} M\otimes Y \xrightarrow{f} N)$ commutes, we get that $\langle m\rangle f= =f\circ(m\otimes Y)$ identifying along the natural adjointness isomorphism.

3.2. Theorem: *Let* $\zeta\colon M(X)\to N(X\otimes Y)$ *be a natural transformation in* X *compatible with the action of* A *on* M *resp.* N. *Then there is a unique* $f\in{}_A[M,N](Y)$ *such that* $\zeta(m)=\langle m\rangle f$ *for all* $X\in\mathscr{C}$ *and* $m\in M(X)$.

PROOF. Without taking into account the A-action, the Yoneda Lemma gives us a unique $\zeta(\mathrm{id})_M=f\in N(M\otimes Y)$ such that $\zeta(m)=f\circ(m\otimes Y)=\langle m\rangle f$ for all $X\in\mathscr{C}$ and all $m\in M(X)$. Now $\zeta(am)=a\zeta(m)$ implies $a(\langle m\rangle f)=\langle am\rangle f$, so that $f\in{}_A[M,N](Y)$.

3.3. Corollary: *There is an associative multiplication*

$${}_A[M,N]\otimes{}_A[N,P]\xrightarrow{\mu}{}_A[M,P]$$

with unit $I\to{}_A[M,M]$ *for all* M, *such that* $(I\otimes{}_A[M,N]\to{}_A[M,M]\otimes{}_A[M,N]\to\to{}_A[M,N])=\lambda$ *and* $({}_A[M,N]\otimes I\to{}_A[M,N]\otimes{}_A[N,N]\to{}_A[M,N])=\varrho$.

PROOF. Define the multiplication by $\langle m\rangle\mu(f\otimes g)=\langle\langle m\rangle f\rangle g$ and write $\mu(f\otimes g)==fg$. This defines a morphism by Proposition 3.2. Then $\langle m\rangle fg=\langle\langle m\rangle f\rangle g$ hence $\langle m\rangle(fg)h=\langle\langle m\rangle fg\rangle h=\langle\langle\langle m\rangle f\rangle g\rangle h=\langle m\rangle f(gh)$ and by Proposition 3.2 we have $(fg)h=f(gh)$, the associativity.

Define $I\xrightarrow{\eta}{}_A[M,M]$ as morphism corresponding to $\varrho\in{}_A\mathscr{C}(M\otimes I,M)$. Then for $m\in M(X)$ we have $\langle m\rangle\eta=\varrho(m\otimes I)=m$ (again identifying X and $X\otimes I$ via ϱ), hence $\eta f=f$ and $g\eta=g$.

3.4. Corollary: *For each* $M\in{}_A\mathscr{C}$ *we get a monoid* ${}_A[M,M]$.

3.5. Corollary: *For* $M\in{}_A\mathscr{C}$, $N\in{}_A\mathscr{C}_B$ *we have* ${}_A[M,N]\in\mathscr{C}_B$.

PROOF. The map

$$M(X)\times{}_A[M,N](Y)\times B(Z)\ni(m,f,b)\mapsto(\langle m\rangle f)b\in N(X\otimes Y\otimes Z)$$

induces a bimorphism

$${}_A[M,N](Y)\times B(Z)\ni(f,b)\mapsto fb\in{}_A[M,N](Y\otimes Z)$$

by Proposition 3.2 such that $(\langle m\rangle f)b=\langle m\rangle(fb)$. Hence ${}_A[M,N]$ becomes a right B-object.

3.6. Corollary: $M\in{}_A\mathscr{C}$ *induces a functor* ${}_A[M,-]\colon{}_A\mathscr{C}_B\to\mathscr{C}_B$.

3.7. Corollary: *Let* $M\in{}_A\mathscr{C}_B$ *and* $N\in{}_A\mathscr{C}$. *Then* ${}_A[M,N]\in{}_B\mathscr{C}$.

PROOF. $M(X)\times B(Y)\times{}_A[M,N](Z)\ni(m,b,f)\mapsto\langle mb\rangle f\in N(X\otimes Y\otimes Z)$ induces a bimorphism

$$B(Y)\times{}_A[M,N](Z)\ni(b,f)\mapsto bf\in{}_A[M,N](Y\otimes Z)$$

by Proposition 3.2 such that $\langle m\rangle(bf)=\langle mb\rangle f$. Hence ${}_A[M,N]$ becomes a left B-object.

3.8. Corollary: $M\in{}_A\mathscr{C}_B$ *induces a functor* ${}_A[M,-]\colon{}_A\mathscr{C}\to{}_B\mathscr{C}$.

3.9. Corollary: *If* $M\in{}_A\mathscr{C}_B$, $N\in{}_A\mathscr{C}_B$ *then* ${}_A[M,N]\in{}_B\mathscr{C}_C$.

3.10. Proposition: *For* $Q\in{}_B\mathscr{C}_A$, $M\in{}_A\mathscr{C}$ *and* $N\in{}_B\mathscr{C}$ *there is a natural isomorphism* ${}_A\mathscr{C}(M, {}_B[Q,N])\cong{}_B\mathscr{C}(Q\otimes_A M, N)$ *if* ${}_B[Q,N]$ *and* $Q\otimes_A M$ *exist.*

PROOF. The difference cokernel in ${}_B\mathscr{C}$ $Q\otimes A\otimes M\rightrightarrows Q\otimes M\to Q\otimes_A M$ induces a commutative diagram of difference kernels

$$\begin{array}{ccccc}
{}_B\mathscr{C}(Q\otimes_A M, N) & \longrightarrow & {}_B\mathscr{C}(Q\otimes M, N) & \rightrightarrows & {}_B\mathscr{C}(Q\otimes A\otimes M, N) \\
\wr\Vert & & \wr\Vert\ \Psi & & \wr\Vert\ \Psi \\
{}_A\mathscr{C}(M, {}_B[Q,N]) & \longrightarrow & \mathscr{C}(M, {}_B[Q,N]) & \rightrightarrows & \mathscr{C}(A\otimes M, {}_B[Q,N])
\end{array}$$

where the left isomorphism results from the universal property of difference kernels.

The only problem is the commutativity of the right lower square (with respect to the multiplication $\nu_Q: Q\otimes A\to Q$). Now for $f\in{}_B\mathscr{C}(Q\otimes M, N)$, $q\in Q(X)$, $a\in A(Y)$ and $m\in M(Z)$ have

$$\langle qa\rangle\Psi(f)(m)=f(qa\otimes m)=f\circ(\nu_Q\otimes M)(q\otimes a\otimes m)=\langle q\rangle\Psi\big(f\circ(\nu_Q\otimes M)\big)(a\otimes m)$$

hence the claimed commutativity.

An object $Q\in{}_B\mathscr{C}_A$ is called A-*coflat* if for all monoids C and objects $M\in{}_A\mathscr{C}_C$ the difference cokernel $Q\otimes_A M\in{}_B\mathscr{C}_A$ exists and if the natural morphism $Q\otimes_A(M\otimes N)\to(Q\otimes_A M)\otimes N$ in ${}_B\mathscr{C}_D$, induced by the associativity of the tensor product, is an isomorphism for $M\in{}_A\mathscr{C}_C$ and $N\in\mathscr{C}_D$.

3.11. Proposition: *Let* $Q\in{}_B\mathscr{C}_A$ *be* A-*coflat. Then there is a natural isomorphism*

$$\Psi: {}_B[Q\otimes_A M, N]\cong{}_A\big[M, {}_B[Q,N]\big]$$

for $M\in{}_A\mathscr{C}$, $N\in{}_B\mathscr{C}$ *and* $Q\in{}_B\mathscr{C}_A$.

PROOF. The isomorphism is given by the following commutative diagram

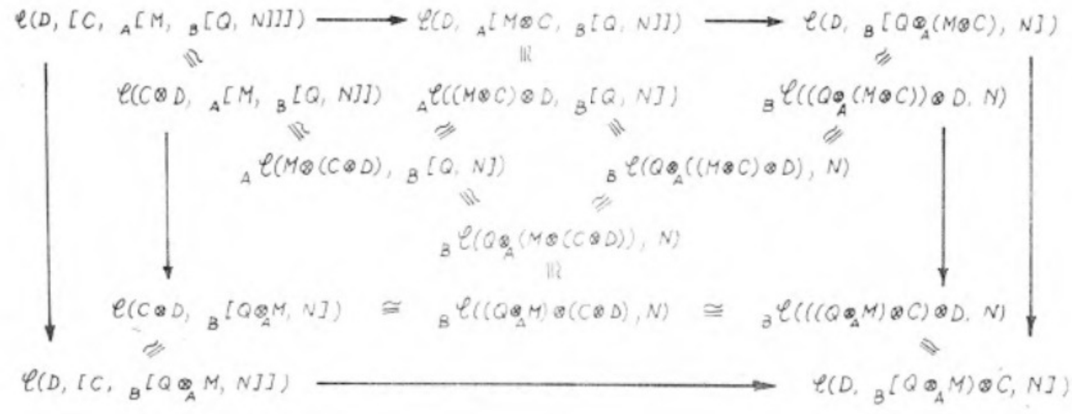
$$\begin{array}{ccccc}
\mathscr{C}(C, {}_B[Q\otimes_A M, N]) & & \xrightarrow{\mathscr{C}(C,\Psi)} & & \mathscr{C}\big(C, {}_A\big[M, {}_B[Q,N]\big]\big) \\
\wr\Vert & & & & \wr\Vert \\
{}_B\mathscr{C}\big((Q\otimes_A M)\otimes C, N\big) & \cong & {}_B\mathscr{C}\big(Q\otimes_A(M\otimes C), N\big) & \cong & {}_A\mathscr{C}(M\otimes C, {}_B[Q,N]).
\end{array}$$

3.12. Proposition: *The following diagram is commutative*

$$\begin{array}{ccccc}
\big[C, {}_A[M, {}_B[Q,N]]\big] & \xleftarrow{\Psi} & {}_A\big[M\otimes C, {}_B[Q,N]\big] & \xleftarrow{\Psi} & {}_B[Q\otimes_A(M\otimes C), N] \\
\uparrow{\scriptstyle [C,\Psi]} & & & & \uparrow{\scriptstyle {}_B[\alpha,N]} \\
\big[C, {}_B[Q\otimes_A M, N]\big] & & \xleftarrow{\qquad\Psi\qquad} & & {}_B[(Q\otimes_A M)\otimes C, N]
\end{array}$$

for $C\in\mathscr{C}$, $M\in{}_A\mathscr{C}$, $N\in{}_B\mathscr{C}$ *and for* $Q\in{}_B\mathscr{C}_A$ A-*coflat.*

PROOF. Invest the definitions of Ψ to get a commutative diagram



The outer frame of this diagram implies the assertion of the proposition.

The examples of § 1 have the following properties. The example a) is closed, complete, cocomplete, and abelian. Example b) is coclosed, complete, cocomplete and abelian. Example c) may be closed (e.g. category of sets with the product), complete, cocomplete and abelian (e.g. K-Mod with the product) or not. Example d) is in general not closed. Example e) is closed, complete, cocomplete and abelian, but not symmetric. Example f) is closed by $[X, Y]$ the set of continuous linear maps [14]. Furthermore there are difference kernels and difference cokernels, but the category is not additive.

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