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Bitopologies and quasi-uniformities on spaces of continuous functions II

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Abstract. We introduce a bitopological notion of pointwise convergence on spaces of continuous functions and show that this notion permits us to extend several classical theorems on function spaces to the bitopological case. In particular, necessary and sufficient conditions for (bicomplete) quasi-metrizability of the bitopology of pointwise convergence are obtained.

1. Introduction

In a recent paper [13] we introduced and investigated bitopological notions of compact convergence and (quasi-)uniform compact convergence on spaces of continuous functions, obtaining several generalizations of classical results on compact convergence and uniform convergence. Almost simultaneously, Papadopoulos [11], [12], and Künzi [6] have also generalized some fundamental results on uniform convergence to the quasi-uniform case from a topological point of view obtaining, among other results, generalizitions of the Ascoli theorem.

We here continue the research begun in [13], introducing and studying the notion of the bitopology of pointwise convergence.

In the following the letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ will denote the set of real numbers, rational numbers and positive integers, respectively. If τ is a

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topology on a set X and $A \subseteq X$, then $\tau \operatorname{cl} A$ ($\tau \operatorname{int} A$) will denote the closure (interior) of A in the topological space (X, τ).

A bitopology on a set X is a pair (τ_1, τ_2) such that each τ_i, $i = 1, 2$, is a topology on X. A bitopological space [5] is an ordered triple (X, τ_1, τ_2) such that X is a set and τ_1 and τ_2 are topologies on X. We say that two bitopologies (τ_1, τ_2) and (τ_1', τ_2') coincide if $\tau_i = \tau_i'$, $i = 1, 2$. Given a bitopology (τ_1, τ_2) we denote by $\tau_1 \vee \tau_2$ the supremum topology of τ_1 and τ_2.

A bitopological space (X, τ_1, τ_2) is called:

(i) 2Hausdorff [15] if $(X, \tau_1, \vee\tau_2)$ is a Hausdorff space.

(ii) pairwise Hausdorff [5] if for $x \neq y$ there is a τ_i-neighborhood of x and a disjoint τ_j-neighborhood of y; $i, j = 1, 2$; $i \neq j$.

(iii) pairwise regular [5] if for each x the τ_j-closed τ_i-neighborhoods of x form a base for the τ_i-neighborhoods of x; $i, j = 1, 2$; $i \neq j$.

(iv) pairwise completely regular [7] if for each x and each τ_i-open set U with $x \in U$ there is a τ_i-lower semicontinuous and τ_j-upper semicontinuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = 0$; $i, j = 1, 2$; $i \neq j$.

(v) 2compact [15] if $(X, \tau_1 \vee \tau_2)$ is compact.

A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that: (i) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subseteq U$ and (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = V \circ V$.

If $\mathcal{U}$ is a quasi-uniformity on X, then $T(\mathcal{U}) = \{A \subseteq X :$ if $x \in A$ there is $U \in \mathcal{U}$ with $U(x) \subseteq A\}$ is a topology on X, where $U(x) = \{y \in X : (x, y) \in U\}$. On the other hand, for each $U \in \mathcal{U}$ we can define $U^{-1} = \{(x, y) : (y, x) \in U\}$. Then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the conjugate of $\mathcal{U}$. The coarsest uniformity finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$ is denoted by $\mathcal{U}^*$. Thus, a basis for $\mathcal{U}^*$ consists of the entourages $U^* = U \cap U^{-1}$ with $U \in \mathcal{U}$. The quasi-uniformity $\mathcal{U}$ is called bicomplete [3] if $\mathcal{U}^*$ is a complete uniformity.

We say that a quasi-uniformity $\mathcal{U}$ on X is compatible with a bitopology (τ_1, τ_2) on X if $T(\mathcal{U}) = \tau_1$ and $T(\mathcal{U}^{-1}) = \tau_2$. A bitopological space (X, τ_1, τ_2) is said to be quasi-uniformizable if there is a quasi-uniformity $\mathcal{U}$ on X compatible with (τ_1, τ_2). Let us recall [7] that a bitopological space is quasi-unifomizable if and only if it is pairwise completely regular.

A quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. d is called separating if $d(x, y) + d(y, x) > 0$ whenever $x \neq y$ and is called a quasi-metric if $d(x, y) > 0$ whenever $x \neq y$ [15].

Each quasi-(pseudo)metric d on X generates a topology $T(d)$ on X which has as a base the family of d-balls $\{B_d(x,r) : x \in X,\ r > 0\}$ where $B_d(x,r) = \{y \in X : d(x,y) < r\}$. Note that the conjugate of d, d^{-1}, given by $d^{-1}(x,y) = d(y,x)$, is also a quasi-(pseudo)metric on X. Thus, a quasi-(pseudo)metric d on X is called compatible with a bitopology (τ_1, τ_2) on X if $T(d) = \tau_1$ and $T(d^{-1}) = \tau_2$. A bitopological space (X, τ_1, τ_2) is called (separated) quasi-(pseudo)metrizable if there is a (separating) quasi-(pseudo)metric on X compatible with (τ_1, τ_2).

Note that a quasi-pseudometric d on X is separating if and only if $(X, T(d),\ T(d^{-1}))$ is a 2Hausdorff space. Similarly, d is a quasi-metric if and only if $(X, T(d),\ T(d^{-1}))$ is pairwise Hausdorff.

If d is a separating quasi-pseudometric on X, then d^* defined by $d^*(x,y) = \max\{d(x,y), d(y,x)\}$, is a metric on X. Furthermore, d is called bicomplete [14] if d^* is complete metric.

Let d be the separating quasi-pseudometric defined on $\mathbb{R}$ by $d(x,y) = \max\{y - x, 0\}$. Then, basic $T(d)$-open sets are of the form $]-\infty, a[$, $a \in \mathbb{R}$, and basic $T(d^{-1})$-open sets are of the form $]a, +\infty[$, $a \in \mathbb{R}$. Note that d^* is the usual metric on $\mathbb{R}$. Therefore d is bicomplete. In the rest of the paper u and ℓ will denote the above topologies $T(d)$ and $T(d^{-1})$, respectively. Note that $(\mathbb{R}, u, \ell)$ is 2Hausdorff but not pairwise Hausdorff. Note also that $([0,1], u, \ell)$ is 2compact.

2. The bitopology of pointwise convergence

Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. A function f from X into Y is called bicontinuous if it is continuous from (X, τ_i) into (Y, τ_i') for $i = 1, 2$. Then we shall write that $f : (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ is bicontinuous.

Following [13] we denote by Y^X the set for all continuous functions from $(X, \tau_1 \vee \tau_2)$ into $(Y, {\tau_1}' \vee {\tau_2}')$ and by BY^X the subset of Y^X which consists of all bicontinuous functions from (X, τ_1, τ_2) into (Y, τ_1', τ_2'). In particular, $BY^{[0,1]}$ will denote the set of all bicontinuous functions from $([0,1], u, \ell)$ into (Y, τ_1', τ_2') and $B[0,1]^X$ the set of all bicontinuous functions from (X, τ_1, τ_2) into $([0,1], u, \ell)$. Similarly we define $BY^{\mathbb{R}}$ and $B\mathbb{R}^X$. By $\mathbb{R}^X$ we denote the set of all continuous functions from $(X, \tau_1 \vee \tau_2)$ into $\mathbb{R}$ with its usual topology $u \vee \ell$.

A topological space (X, τ) contains a nontrivial path provided that there is a continuous function $p : ([0,1], u \vee \ell) \to (X, \tau)$ such that $p(0) \neq p(1)$. In the following by "path" we mean a nontrivial path. We say that a bitopological space (Y, τ_1', τ_2') contains a pairwise path [13] provided that there is a bicontinuous function $p : ([0,1], u, \ell) \to (Y, \tau_1', \tau_2')$ such that $p(0) \in Y \setminus \tau_1' \operatorname{cl} p(1)$ and $p(1) \in Y \setminus \tau_2' \operatorname{cl} p(0)$.

We say that (Y, τ_1', τ_2') contains a *2path* provided that there is a path $p : ([0,1], u \vee \ell) \to (Y, \tau_1' \vee \tau_2')$. It is clear that every pairwise path is a 2path.

Given two bitopological spaces (X, τ_1, τ_2) and (Y, τ_1', τ_2') let $\mathcal{K} = \{K \subseteq X : K \text{ is } \tau_1 \vee \tau_2\text{-compact}\}$. For each $K \in \mathcal{K}$ and each $G_i \in \tau_i'$, $i = 1, 2$, consider the set

$$[K, G_i] = \{f \in Y^X : f(K) \subseteq G_i\}$$

Then $\{[K, G_i] : K \in \mathcal{K} \text{ and } G_i \in \tau_i'\}$ is a subbase for a topology T_k^i, $i = 1, 2$, on Y^X. By analogy wih the topological case, (T_k^1, T_k^2) will be called the 2compact open bitopology (on Y^X) [13]. In the following that bitopology will be called *the bitopology of 2compact convergence* (*on* Y^X).

Now let $\mathcal{F}$ be the subset of $\mathcal{K}$ which consists of all (nonempty) finite subsets of X. Then $\{[F, G_i] : F \in \mathcal{F} \text{ and } G_i \in \tau_i'\}$ is a subbase for a topology T_p^i, $i = 1, 2$, on Y^X. The bitopology (T_p^1, T_p^2) will be called *the bitopology of pointwise convergence* (on Y^X). The subset BY^X of Y^X endowed with the restriction of this bitopology will be denoted by (BY^X, T_p^1, T_p^2).

Remark 1. [13, Example 2] shows that in general the topologies T_p^1 and T_p^2 are not comparable.

Remark 2. Given the bitopological spaces (Y^X, T_k^1, T_k^2) and (Y^X, T_p^1, T_p^2), then $T_p^i \subseteq T_k^i$, $i = 1, 2$.

Remark 3. It is shown in [13, Remark 2] that $B\mathbb{R}^X$ is dense in $(\mathbb{R}^X, T_k^i)$, $i = 1, 2$. It follows from Remark 2, that $B\mathbb{R}^X$ is also dense in $(\mathbb{R}^X, T_p^i)$, $i = 1, 2$.

Related to Remark 1 we may state the two following results on comparison of topologies. With the notation introduced above, we have:

Proposition 1. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. Then, $\tau_1' \subseteq \tau_2'$ if and only if $T_k^1 \subseteq T_k^2$.*

Proof. Suppose $\tau_1' \subseteq \tau_2'$. Each subbasic open set $[K, G]$ of T_k^1 (with $K \in \mathcal{K}$ and $G \in \tau_1'$) is also open in T_k^2 because $G \in \tau_2'$. Thus $T_k^1 \subseteq T_k^2$. Conversely, let $y \in Y$ and let G be a τ_1'-open neighborhood of y. Define $f_y : X \to Y$ by $f_y(x) = y$ for all $x \in X$. Fix $x_0 \in X$. Then $f_y \in [x_0, G] \in T_k^1$. Hence, there exist $\tau_1' \vee \tau_2'$-compact non-void sets $K_1, \ldots, K_n$, and τ_2' -open sets $G_1, \ldots, G_n$, such that $f_y \in \bigcap\{[K_i, G_i] : i = 1, \ldots, n\} \subseteq [x_0, G]$.

Then it is easy to see that $y \in \bigcap\{G_i : i = 1, 2, \ldots, n\} \subseteq G$. (If $z \in \bigcap\{G_i : i = 1, 2, \ldots, n\}$, then $f_z \in \bigcap\{[K_i, G_i] : i = 1, 2, \ldots, n\}$, so that $z \in G$.) We conclude that $\tau_1' \subseteq \tau_2'$.

If in the proof of the preceding result, $\tau_1 \vee \tau_2$-compact sets are replaced by singletons, we obtain the following

Propsition 2. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. Then, $\tau_1' \subseteq \tau_2'$ if and only if $T_p^1 \subseteq T_p^2$.*

The next result will be useful in Sections 3 and 4 (compare with [13, Lemma 3]).

Lemma 1. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. If we denote by T_p^* the topology of pointwise convergence on Y^X relative to $(X, \tau_1 \vee \tau_2)$ and $(Y, \tau_1' \vee \tau_2')$, then $T_p^* = T_p^1 \vee T_p^2$.*

Proof. Let $f \in Y^X$, F a (nonempty) finite subset of X and W a $\tau_1' \vee \tau_2'$-open subset of Y such that $f \in [F, W]$. For each $a \in F$ there exist a τ_1'-open set G_a and a τ_2'-open set H_a with $f(a) \in G_a \cap H_a \subseteq W$. Set $F_a = F \cap f^{-1}(G_a \cap H_a)$ whenever $a \in F$. Putting $L = \bigcap\{[F_a, G_a] : a \in F\}$ and $M = \bigcap\{[F_a, H_a] : a \in F\}$ we have that $L \in T_p^1$, $M \in T_p^2$ and $f \in L \cap M$. Since $L \cap M \subseteq [F, W]$, $T_p^* \subseteq T_p^1 \vee T_p^2$. Finally, the inclusion $T_p^1 \vee T_p^2 \subseteq T_p^*$ is obvious.

It is well-known that if (X, τ) and (Y, τ') are two topological spaces, then the topology of pointwise convergence coincides with the topology of a subspace of the Cartesian product $\prod_{x \in X} Y_x$ where $Y_x = Y$ for all $x \in X$. Similarly, we obtain the following bitopological generalization.

Proposition 3. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. Then the bitopology of pointwise convergence coincides with the bitopology of a subspace of the product space $\left(\prod_{x \in X} Y_x, \prod_{x \in X} (\tau_1')_x, \prod_{x \in X} (\tau_2')_x\right)$, where $Y_x = Y$, $(\tau_1')_x = \tau_1'$ and $(\tau_2')_x = \tau_2'$ for all $x \in X$.*

From the above result and [16, Theorems 1.3, 1.4 and 1.5] we deduce the following corollary.

Corollary. *Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces. Then:*

(a) *(Y^X, T_p^1, T_p^2) is 2Hausdorff if and only if (Y, τ_1, τ_2) is 2Hausdorff.*

(b) *(Y^X, T_p^1, T_p^2) is pairwise Hausdorff if and only if $(Y, {\tau_1}', {\tau_2}')$ is pairwise Hausdorff.*

(c) (Y^X, T_p^1, T_p^2) *is pairwise regular if and only if* (Y, τ_1', τ_2') *is pairwise regular.*

(d) (Y^X, T_p^1, T_p^2) *is quasi-uniformizable if and only if* (Y, τ_1', τ_2') *is quasi-uniformizable.*

3. The bitopology of quasi-uniform pointwise convergence

Let (X, τ_1, τ_2) and (Y, τ_1', τ_2') be two bitopological spaces such that (Y, τ_1', τ_2') is quasi-uniformizable and let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1', τ_2'). Then the collection of sets of the form $(K, U) = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U \text{ for all } x \in K\}$, where $K \in \mathcal{K}$ and $U \in \mathcal{U}$, is a base for a quasi-uniformity $\mathcal{U}_k$ on Y^X called the quasi-uniformity of quasi-uniform convergence (of $\mathcal{U}$) on 2compacta [13] (see also [9] and [11]). The bitopology $(T(\mathcal{U}_k), T(\mathcal{U}_k^{-1}))$ is said to be the bitopology of quasi-uniform convergence (of $\mathcal{U}$) on 2compacta [13].

Similarly, the collection of sets of the form $(X, U)=\{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U \text{ for all } x \in X\}$, where $U \in \mathcal{U}$, is a base for a quasi-uniformity $\mathcal{U}_X$ on Y^X called the quasi-uniformity of quasi-uniform convergence (of $\mathcal{U}$) [13] (see also [9] and [12]). The bitopology $(T(\mathcal{U}_X), T(\mathcal{U}_X^{-1}))$ is said to be the bitopology of quasi-uniform convergence (of $\mathcal{U}$) [13].

Now consider the collection of sets of the form

$$(x, U) = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in U\}$$

where $x \in X$ and $U \in \mathcal{U}$. Then, this collection is a subbase for a quasi-uniformity $\mathcal{U}_p$ on Y^X called *the quasi-uniformity of pointwise convergence* (*of* $\mathcal{U}$) (see [11]). The bitopology $(T(\mathcal{U}_p), T(\mathcal{U}_p^{-1}))$ is said to be *the bitopology of quasi-uniform pointwise convergence* (*of* $\mathcal{U}$).

Remark 4. The following inclusions are evident: $\mathcal{U}_p \subseteq \mathcal{U}_k \subseteq \mathcal{U}_X$.

It is shown in [13] that if the quasi-unifomity $\mathcal{U}$ is bicomplete, then $\mathcal{U}_X$ is bicomplete. If, in addition, $(X, \tau_1 \vee \tau_2)$ is a k-space, then $\mathcal{U}_k$ is also bicomplete ([13, Lemma 1] and [4, Chapter 7, Theorem 12]). Now we obtain the corresponding result on bicompleteness of the quasi-uniformity of pointwise convergence.

Proposition 4. *Let (X,τ_1,τ_2) be a bitopological space such that $\tau_1 \vee \tau_2$ is the discrete topology on X and let (Y,τ_1',τ_2') be a quasi-uniformizable space. If (τ_1',τ_2') has a compatible bicomplete quasi-uniformity $\mathcal{U}$, then $\mathcal{U}_p$ is bicomplete.*

Proof. Since each function $f : X \to Y$ is in Y^X, the quasi-uniform space $(Y^X,\mathcal{U}_p)$ coincides with the quasi-uniform product $\prod_x (Y,\mathcal{U})$. The latter is bicomplete since $(Y,\mathcal{U})$ is bicomplete.

Our next results generalize several fundamental facts on the topology of pointwise convergence to the bitopological case.

Lemma 2. *Let (X,τ_1,τ_2) be a bitopological space and (Y,τ_1',τ_2') a quasi-uniformizable space. Then for each quasi-uniformity on Y compatible with (τ_1',τ_2'), the bitopology of quasi-uniform pointwise convergence coincides with the bitopology of pointwise convergence.*

Proof. Let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1',τ_2'). We first show that $T_p^1 \subseteq T(\mathcal{U}_p)$. Let $f \in Y^X$, F a (nonempty) finite subset of X and $G \in \tau_1'$ such that $f \in [F,G]$. Then there is $U \in \mathcal{U}$ such that $U(f(x)) \subseteq G$ for all $x \in F$. Set $V = \bigcap\{(x,U) : x \in F\}$. Then $V \in \mathcal{U}_p$ and $V(f) \subseteq [F,G]$. Thus $T_p^1 \subseteq T(\mathcal{U}_p)$. Similarly we show that $T_p^2 \subseteq T(\mathcal{U}_p^{-1})$. Conversely, let $f \in Y^X$, $x_0 \in X$ and $U \in \mathcal{U}$, then $f \in [x_0,G] \subseteq (x_0,U)(f)$ where $G = \tau_1' \operatorname{int} U(f(x_0))$, so that $T(\mathcal{U}_p) \subseteq T_p^1$. Similarly we show that $T(\mathcal{U}_p^{-1}) \subseteq T_p^2$.

Theorem 1. *Let (X,τ_1,τ_2) be a 2Hausdorff quasi-uniformizable space and (Y,τ_1',τ_2') a quasi-uniformizable space containing a pairwise path. Then:*

(a) *The bitopology of 2compact convergence coincides with the bitopology of pointwise convergence if and only if every $\tau_1 \vee \tau_2$-compact subset of X is finite.*

(b) *For each quasi-uniformity on Y compatible with (τ_1',τ_2'), the bitopology of quasi-uniform convergence coincides with the bitopology of pointwise convergence if and only if X is a finite set.*

Proof. (a) Suppose that the bitopology of 2compact convergence coincides with the bitopology of pointwise convergence. By Lemma 1 and [13, Lemma 3], the topology of compact convergence and the topology of pointwise convergence (relative to $(X,\tau_1 \vee \tau_2)$ and $(Y,\tau_1' \vee \tau_2')$) coincide.

Hence every $\tau_1 \vee \tau_2$-compact subset of X is finite (see [8, proof of Theorem 1.1.1] and [8, Theorem 1.1.4]). The converse is obvious.

(b) Suppose that X is a finite set. Then, it is clear that for each quasi-uniformity on Y compatible wih (τ_1', τ_2') the bitopology of quasi-uniform convergence coincides with the bitopology of quasi-uniform pointwise convergence and, by Lemma 2, it coincides wih the bitopology of pointwise convergence. Conversely, let $\mathcal{U}$ be a quasi-uniformity on Y compatible with (τ_1', τ_2'). Assume that the bitopology of quasi-uniform convergence coincides with the bitopology of pointwise convergence. It follows from Remark 4 and [13, Lemma 2] that the bitopology of qasi-uniform convergence coincides with the bitopology of 2compact convergence. By the proof of [13, Theorem 2], (X, τ_1, τ_2) is 2compact. Since the bitopology of 2compact covergence also coincides with the bitopology of pointwise convergence, every $\tau_1 \vee \tau_2$-compact subset of X is finite by (a). Therefore X is finite.

The following examples illustrate the results established in the above theorem.

Example 1. Let $X = \mathbb{N}$, τ_1 the cofinite topology on X and τ_2 the discrete topology on X, and let $Y = \mathbb{R}$, $\tau_1' = u$ and $\tau_2' = \ell$. Then the bitopology of pointwise convergence coincides with the bitopology of 2compact convergence by Theorem 1 (a).

Example 2. Let $X = \{0, 1\}$, τ_1 the Sierpinski topology (i.e., $\tau_1 = \{\emptyset, \{1\}, X\}$) and $\tau_2 = \{\emptyset, \{0\}, X\}$, and let $Y = \mathbb{R}$, $\tau_1' = u$ and $\tau_2' = \ell$. By Theorem 1 (b) for each quasi-uniformity on Y compatible with (u, ℓ), the bitopology of quasi-uniform convergence coincides with the bitopology of pointwise convergence.

4. Quasi-pseudometrizability of the bitopology of pointwise convergence

It is well-known (see, for instance, [8]) that if (X, τ) is a Tychonoff space and (Y, τ') is a space containing a path, then the following theorems hold:

Theorem A. *The topology of compact convergence is metrizable if and only if (X, τ) is hemicompact and (Y, τ') is metrizable.*

Theorem B. *The topology of compact convergence is completely metrizable if and only if (X, τ) is a hemicompact k-space and (Y, τ') is completely metrizable.*

Therem C. *The topology of pointwise convergence is metrizable if and only if X is countable and (Y, τ') is metrizable.*

Theorem D. *The topology of pointwise convergence is completely metrizable if and only if (X, τ) is countable and discrete and (Y, τ') is completely metrizable.*

In [13] we prove the following bitopological generalizations of Theorems A and B, respectively:

Theorem A'. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a bitopological space containing a pairwise path. Then the bitopology of 2compact convergence is separated quasi-pseudometrizable if and only if $(X, \tau_1 \vee \tau_2)$ is a hemicompact space and (Y, τ_1', τ_2') is separated quasi-pseudometrizable.*

Theorem B'. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a bitopological space containing a pairwise path. Then the bitopology of 2compact convergence is bicompletely separated quasi-pseudometrizable if and only if $(X, \tau_1 \vee \tau_2)$ is a hemicompact k-space and (Y, τ_1', τ_2') is a bicompletely separated quasi-pseudometrizable space.*

Remark 5. Note that Theorems A' and B' remain true if "pairwise path" is replaced by "2path". Note also [13, Remark 4] that such a condition is only used in the proof of the forward implications.

Now, we shall extend the results on (complete) metrizability of the topology of pointwise convergence (Theorems C and D) to the bitopological case.

Let us recall that a bitopological space is said to be bicompletely (separated) quasi-(pseudo)metrizable if it has a compatible bicomplete (separating) quasi-(pseudo)metric.

Theorem 2. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a bitopological space containing a 2path. Then the bitopology of pointwise convergence is separated quasi-pseudometrizable if and only if X is countable and (Y, τ_1', τ_2') is separated quasi-pseudometrizable.*

Proof. Necessity. Let d be a separating quasi-pseudometric on Y^X compatible with (T_p^1, T_p^2). For each $y \in Y$ let $f_y : X \to Y$ be defined by $f_y(x) = y$ for all $x \in X$. Then $f_y \in Y^X$. Now define for each

$y, z \in Y$, $\rho(y,z) = d(f_y, f_z)$. It is easily seen that ρ is a separating quasi-pseudometric on Y compatible wih (τ_1', τ_2'). Since d^* is a metric on Y^X compatible with $T_p^1 \vee T_p^2$, it follows from Lemma 1 and Theorem C that X is countable.

Sufficiency. Since (Y, τ_1', τ_2') is quasi-pseudometrizable and the countable product of (separated) quasi-pseudometrizable bitopological spaces is (separated) quasi-pseudometrizable, it follows from Proposition 3 that (Y^X, T_p^1, T_p^2) is a subspace of a separated quasi-pseudometrizable bitopological space, hence it is separated qasi-pseudometrizable.

Lemma 3 [14, Theorem 2.1]. *A quasi-pseudometrizable bitopological space (X, τ_1, τ_2) is bicompletely quasi-pseudometrizable if and only if $(X, \tau_1 \vee \tau_2)$ is completely pseudometrizable.*

Theorem 3. *Let (X, τ_1, τ_2) be a 2Hausdorff quasi-uniformizable space and (Y, τ_1', τ_2') a bitopological space containing a 2path. Then the bitopology of pointwise convergence is bicompletely separated quasi-pseudometrizable if and only if $(X, \tau_1 \vee \tau_2)$ is countabe and discrete and (Y, τ_1', τ_2') is bicompletely separated quasi-pseudometrizable.*

Proof. Necessity. By Theorem 2, X is countable and (Y, τ_1', τ_2') is separated quasi-pseudometrizable. Since $T_p^1 \vee T_p^2$ is a completely metrizable topology and, by Lemma 1, that topology is exactly the topology of pointwise convergence relative to $(X, \tau_1 \vee \tau_2)$ and $(Y, \tau_1' \vee \tau_2')$, it follows from Theorem D that $(X, \tau_1 \vee \tau_2)$ is discrete and $(Y, \tau_1' \vee \tau_2')$ is completely metrizable, so that (Y, τ_1', τ_2') is bicompletely separated quasi-pseudometrizable by Lemma 3.

Sufficiency. Since $(X, \tau_1 \vee \tau_2)$ is countable and discrete and $(Y, \tau_1' \vee \tau_2')$ is completely metrizable, it follows from Theorem D and Lemma 1 that $(Y^X, T_p^1 \vee T_p^2)$ is completely metrizable. Since, by Theorem 2, (Y^X, T_p^1, T_p^2) is separated quasi-pseudometrizable, we obtain the result applying Lemma 3.

Remark 6. Note that in Theorems 2 and 3, the condition that (Y, τ_1', τ_2') contains a 2path is only used in the proof of the forward implication.

Example 3. Let Y be the set of all nonnegative reals, let τ'_1 be the usual topology on Y and let $\tau'_2 = \{\emptyset\} \cup \{G \cup]x, +\infty[\, : G \in \tau'_1 \text{ and } x \in Y\}$. Then $\tau'_2 \subset \tau'_1$, $\tau'_2 \neq \tau'_1$ and (Y, τ'_1, τ'_2) is a pairwise Hausdorff pairwise compact space (see [2, Example 4]). By [2, Theorem 12] it is pairwise regular. Since both τ'_1 and τ'_2 have a countable base and (Y, τ'_1) is completely metrizable it follows from [5, Theorem 2.8] and Lemma 3 that (Y, τ'_1, τ'_2) is bicompletely quasi-metrizable. Furthermore, it contains a 2path but not a pairwise path because both τ'_1 and τ'_2 are T_1 topologies (in fact, it is easy to show that if a bitopological space (Y, τ'_1, τ'_2) contains a pairwise path then τ'_i is not a T_1 topology, $i = 1, 2$). Now let $X = Y$ and $\tau_1 = \tau_2$ the usual topology on Y. By Theorem B' and Remark 5, the bitopology (T^1_k, T^2_k) of 2compact convergence is bicompletely quasi-metrizable. Note that, by Proposition 1, $T^1_k \subset T^2_k$ and $T^1_k \neq T^2_k$ and, by Theorem 2, the bitopology of pointwise convergence is not quasi-metrizable.

If now we put $X = Y \cap \mathbb{Q}$ and $\tau_1 = \tau_2$ the usual topology on X with again (Y, τ'_1, τ'_2) as above, then the bitopology of 2compact convergence is bicompletely quasi-metrizable and the bitopology of pointwise convergence is quasi-metrizable but not bicompletely quasi-metrizable. Again we have $T^1_k \subset T^2_k$, $T^1_p \subset T^2_p$, $T^1_k \neq T^2_k$ and $T^1_p \neq T^2_p$.

It is clear that if (Y^X, T^1_p, T^2_p) is separated quasi-pseudometrizable, so is (BY^X, T^1_p, T^2_p). We do not know if the converse holds. However, that converse holds whenever (X, τ_1, τ_2) is pairwise Hausdorff.

Proposition 5. *Let (X, τ_1, τ_2) be a pairwise Hausdorff quasi-uniformizable space and (Y, τ'_1, τ'_2) a bitopological space containing a pairwise path. Then the following are equivalent:*

(1) *X is countable and (Y, τ'_1, τ'_2) is separated quasi-pseudometrizable.*

(2) *(Y^X, T^1_p, T^2_p) is separated quasi-pseudometrizable.*

(3) *(BY^X, T^1_p, T^2_p) is separated quasi-pseudometrizable.*

Proof. (1) $\Longrightarrow$ (2). Theorem 2.

(2) $\Longrightarrow$ (3). Obvious.

(3) $\Longrightarrow$ (1). Let d be a separating quasi-pseudometric on BY^X compatible with (T^1_p, T^2_p). For each $y \in Y$ define $f_y : X \to Y$ by $f_y(x) = y$ for all $x \in X$. Since $f_y \in BY^X$, it easily follows that the real valued function ρ defined on $Y \times Y$ by $\rho(y, z) = d(f_y, f_z)$ is a separating quasi-pseudometric compatible with (τ'_1, τ'_2). It remains to show that X is a countable set. To this end, let p be a pairwise path for (Y, τ'_1, τ'_2). Define $f : X \to Y$ by $f(x) = p(0)$ for all $x \in X$. Clearly, $f \in BY^X$. Since f is

a constant function, for each $n \in \mathbb{N}$ there is a finite set $F_n \subseteq X$ and a τ_1'-open set G_n such that $f \in [F_n, G_n] \subseteq B_d(f, 2^{-n})$. We want to show that $X = \bigcup\{F_n : n \in \mathbb{N}\}$. Suppose that there exists $x_0 \in X \setminus \bigcup\{F_n : n \in \mathbb{N}\}$. Since (Y, τ_1', τ_2') is pairwise completely regular and τ_2 is a T_1 topology, then for each $n \in \mathbb{N}$ there is $g_n \in B[0,1]^X$ such that $g_n(F_n) = 0$ and $g_n(x_0) = 1 - (1/(n+1))$. Thus, $p \circ g_n \in BY^X$ for all $n \in \mathbb{N}$. Furthermore, $p \circ g_n \in [F_n, G_n] \subseteq B_d(f, 2^{-n})$ for all $n \in \mathbb{N}$, so that sequence $\langle p \circ g_n \rangle$ converges to f with respect to T_p^1. On the other hand, since p is a pairwise path, there is a τ_1'-open neighborhood G of $p(0)$ such that $p(1) \in Y \setminus G$. Therefore the u-open set $p^{-1}(G)$ is of the form $[0, \delta[$ for some δ such that $0 < \delta < 1$. Thus, there is $k \in \mathbb{N}$ such that $p \circ g_n \in BY^X \setminus [x_0, G]$ for all $n \geq k$. Since $f \in [x_0, G] \in T_p^1$, we conclude that the sequence $\langle p \circ g_n \rangle$ does not converge to f with respect to T_p^1. This contradiction concludes the proof.

Problem. In view of the above result the following open question may be of some interest in this context: Can one replace "pairwise path" by "2path" in the statement of Proposition 5?

We conclude this section with some more examples on (bicomplete) quasi-metrizability of the bitopology of pointwise convergence.

Example 4. Let $X = \mathbb{Q}$, τ_1 the Sorgenfrey topology on $\mathbb{Q}$ (τ_1-basic open sets are of the form $[x, y[$ in $\mathbb{Q}$, $x < y$) and τ_2 the Sorgenfrey conjugate topology on $\mathbb{Q}$ (τ_2-basic open sets are of the form $]x, y]$ in $\mathbb{Q}$, $x < y$). Let $Y = \mathbb{R}$, $\tau_1' = u$ and $\tau_2' = \ell$. Then the bitopology of pointwise convergence is bicompletely separated quasi-pseudometrizable as Theorem 3 shows.

Example 5. Let $X + \mathbb{Q}$, $\tau_1 = U \mid \mathbb{Q}$ and $\tau_2 = \ell \mid \mathbb{Q}$. Let $Y = \mathbb{R}$, $\tau_1' = u$ and $\tau_2' = \ell$. Then the bitopology of pointwise convergence is separated quasi-pseudometrizable but not bicompletely (separated) quasi-pseudometrizable as Theorems 2 and 3 show.

Example 6. Let X be a countable set, τ_1 any T_1 topology on X and τ_2 the discrete topology on X. Let $Y = \mathbb{R}$, τ_1' the Sorgenfrey topology on $\mathbb{R}$ and τ_2' the Sorgenfrey conjugate topology. Then the bitopology of pointwise convergence is bicompletely quasi-metrizable by Theorem 3 and Remark 6.

5. Quasi-pseudometrizability of bitopologies on $\mathbb{R}^X$

In this section we shall apply the above results to obtain quasi-pseudometrization theorems in the special and interesting case that the space of continuous functions is $\mathbb{R}^X$.

Let us recall that if (X, τ_1, τ_2) is any bitopological space, $\mathbb{R}^X$ consists of all continuous functions from the topological space $(X, \tau_1 \vee \tau_2)$ into $\mathbb{R}$ with its usual topology $u \vee \ell$. (Note that the embedding $([0,1], u, \ell) \to (\mathbb{R}, u, \ell)$ is a pairwise path, hence a 2path.)

Theorem 4. *For a 2Hausdorff quasi-uniformizable space (X, τ_1, τ_2) the following are equivalent:*

(1) $(\mathbb{R}^X, T_k^1, T_k^2)$ *is separated quasi-pseudometrizable.*

(2) $(\mathbb{R}^X, T_k^i)$ *is first countable for some $i = 1, 2$.*

(3) $(X, \tau_1 \vee \tau_2)$ *is hemicompact.*

Proof. (1) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (3). Suppose, for instance, that $(\mathbb{R}^X, T_k^1)$ is first countable. Consider the function $f : X \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in X$. Then $f \in \mathbb{R}^X$. Let $\{U_n : n \in \mathbb{N}\}$ be a base of T_k^1-open neighborhoods of f. Then, there are a sequence $\langle K_n \rangle$ of $\tau_1 \vee \tau_2$-compact subsets of X and a descreasing sequence $\langle \delta_n \rangle$ of positive real numbers such that $\delta_n \to 0$ and $f \in [K_n, (-\infty, \delta_n)] \subseteq U_n$. Given $K \in \mathcal{K}$ set $A = [K, (-\infty, 1)]$. Hence, there is U_n such that $f \in U_n \subseteq A$. We shall show that $K \subseteq K_n$. Assume the contrary. Then, there is $x \in K \setminus K_n$, so that there is $\Phi \in [0,1]^X$ such that $\Phi(x) = 1$ and $\Phi(K_n) = 0$. Thus $\Phi \in [K_n, (-\infty, \delta_n)] \subseteq U_n \subseteq A$, but $\Phi(x) = 1$ implies $\Phi \in \mathbb{R}^X \setminus A$, a contradiction.

(3) $\Longrightarrow$ (1). Apply Theorem A'.

Theorem 5. *For a 2Hausdorff quasi-uniformizable space (X, τ_1, τ_2) the following are equivalent:*

(1) $(\mathbb{R}^X, T_k^1, T_k^2)$ *is bicompletely separated quasi-pseudometrizable.*

(2) $(\mathbb{R}^X, T_k^1 \vee T_k^2)$ *is Čech complete.*

(3) $(X, \tau_1 \vee \tau_2)$ *is a hemicompact k-space.*

Proof. (1) $\Longrightarrow$ (2). $(\mathbb{R}^X, T_k^1 \vee T_k^2)$ is completely metrizable and, hence, Čech complete.

(2) $\Longrightarrow$ (3). Apply [13, Lemma 3] and [8, Corollary 5.2.2].

(3) $\Longrightarrow$ (1). Apply Theorem B'.

Theorem 6. *For a 2Hausdorff quasi-uniformizable space* (X, τ_1, τ_2) *the following are equivalent:*

(1) $(\mathbb{R}^X, T_p^1, T_p^2)$ *is separated quasi-pseudometrizable.*

(2) $(\mathbb{R}^X, T_p^i)$ *is first countable for some* $i = 1, 2$.

(3) X *is countable.*

(4) $(\mathbb{R}^X, T_p^1 \vee T_p^2)$ *is second countable.*

(5) $(\mathbb{R}^X, T_p^i)$ *is second countable for some* $i = 1, 2$.

Proof. (1) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (3). Suppose, for instance, that $(\mathbb{R}^X, T_p^1)$ is first countable. Consider the function $f : X \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in X$. Then a slight modification of the proof of Proposition 5 permits us to conclude that X is countable.

(3) $\Longrightarrow$ (1). Apply Theorem 2.

(3) $\Longrightarrow$ (4). Apply Lemma 1 and [1, Theorem I.3.7.].

(4) $\Longrightarrow$ (5). If $(\mathbb{R}^X, T_p^1 \vee T_p^2)$ is second countable, then it has a countable dense subset $\{f_n : n \in \mathbb{N}\}$. Furthermore, X is countable by Lemma 1 and Theorem C. It follows from Theorem 2 that $(\mathbb{R}^X, T_p^1, T_p^2)$ is a separated quasi-pseudometrizable space. Let d be a quasi-pseudometric on $\mathbb{R}^X$ compatible with (T_p^1, T_p^2). It is easily seen that $\{B_d(f_n, 2^{-m}) : n, m \in \mathbb{N}\}$ is a base for T_p^1. (Similarly, $\{B_d^{-1}(f_n, 2^{-m}) : n, m \in \mathbb{N}\}$ is base for T_p^2.)

(5) $\Longrightarrow$ (2). Obvious.

Theorem 7. *For a 2Hausdorff quasi-unifomizable space* (X, τ_1, τ_2) *the following are equivalent:*

(1) $(\mathbb{R}^X, T_p^1, T_p^2)$ *is bicompletely separated quasi-pseudometrizable.*

(2) $(X, \tau_1 \vee \tau_2)$ *is countable and discrete.*

(3) $(\mathbb{R}^X, T_p^1 \vee T_P^2)$ *is Čech complete.*

Proof. (1) $\Longleftrightarrow$ (2). Apply Teorem 3.
(2) $\Longleftrightarrow$ (3). Apply Lemma 1 and [1, Corollary I.3.3].

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