A class of maps acting on semigroups

By M. SATYANARAYANA (Bowling Green, Ohio)

In this note we shall introduce a new class of maps called vector maps and show that the structure and the existence of vector maps can be used to characterize some types of semigroups, especially idempotent semigroups and left or right zero semigroup.*)

A single valued function F from a semigroup S to S is called a vector map if $(xy)F=(xF)y$, $x, y\in S$ and $(xF)F=(xF)(xF)$ for every $x\in S$. If a semigroup S has a zero then the map which sends every element into zero is clearly a vector map. The identity map on idempotent semigroup is a vector map. But infinite cyclic semigroups and the multiplicative semigroup of real numbers without 0 do not possess vector maps. Now we shall prove the existence of vector maps is equivalent to the presence if a zero in case the semigroup has a unique idempotent. This result is not true if the semigroup has more than one idempotent. The structure and the existence of vector maps can be used to characterize some types of semigroups, especially idempotent semigroups and right or left zero semigroups. The results we prove in this direction supplement the results of TAMURA, one of which states a semigroup is a right zero semigroup iff the only left translation is the identity map [2]. We refer the reader to the concepts undefined here to Clifford's book [1]. A single valued function F is called an inner left translation of a semigroup S if $F: x\to ax$, $x\in S$ and we shall denote this by a_l.

1. *Inner left translations*

Evidently vector maps are left translations by virtue of our definition. But the identity map on an infinite cyclic semigroup is a left translation but not a vector map, which can be seen in the following basic result.

1.1 Lemma. *If F is a vector map on a semigroup S and $xF=y$ then $y^3=y^4$ and hence y^3 is an idempotent.*

*) AMS 1970 subject classification: 20M15.

Key words: vector maps, inner left translations, identity map, idempotent semigroups, right zero semigroups.

PROOF. Since $(xF)F=(xF)^2$, $yF=y^2$. So $y^2F=(yF)y=y^3$. Thus $y^nF=y^{n+1}$ for every natural number n. Hence $y^4=(yF)^2=(yF)F=y^2F=y^3$. Clearly y^3 is an idempotent.

An easy verification yields the following

1.2 Theorem. *For a semigroup S the following are equivalent:*

1. *S is an idempotent semigroup.*
2. *Identity map is a vector map.*

In this case every left translation is vector map.

All the three conditions in 1.2 are equivalent if the semigroup has an identity since the identity map is an inner left translation in the presence of an identity. However if the semigroup has no identity it need not be an idempotent semigroup though every inner left translation is a vector map. The semigroup $S=\{x, x^2: x^2=x^3\}$ provides the necessary example.

1.3 Theorem. *In a semigroup S every inner left translation is a vector map iff $x^2a=(xa)^2$ and $a^2x=(ax)^2$ for every a and x in S. Also if $x\in S$, x^3 is an idempotent and $x^3=x^4$ and the set E of all idempotents is a right ideal. Furthermore S is an extension of SE by a semigroup in which every element is nilpotent of index 3. In particular if S is commutative, then S is an extension of an idempotent semigroup by a semigroup in which every element is nilpotent of index 3.*

PROOF. Let $x\in S$. If x_l is a vector map and if $a\in S$, then $a(x_l o x_l)=(ax_l)^2$ and so $x^2a=(xa)^2$. Set $x=a$. Then $x^3=x^4$ for every $x\in S$. Thus x^3 is an idempotent. Also if x is an idempotent, then $xa=(xa)^2$ for every $a\in S$. Hence E is a right ideal. Let $x\notin SE$. Since $x^3\in E$, $x^3=x^4\in SE$. Hence S in an extension of SE by a semigroup in which every element is nilpotent of index 3. In commutative case clearly $SE=E$. Thus the last result follows.

However the converse of 1.3 is not true. Let S be a commutative semigroup generated by e, x and y subject to the conditions $e=xe$; $e=ye$; $x^3=x^4$; $y^3=y^4$; $x^3y=x^3y^3$ and $y^3x=y^3x^2$. x_l is not a vector map since $y(x_l o x_l)\neq(yx_l)^2$.

It can be easily verified that an identity map on a semigroup S is an inner left translation iff S has a left identity. Under composition the set $I(S)$ of all inner left translations on a semigroup S is again a semigroup. $I(S)$ may have an identity but this might not necessarily be the identity map. For, the semigroup $S=\{x, x^2: x^2= =x^3\}$, x_l is the identity of $I(S)$ but not an identity map. Now we shall characterize those semigroups with $I(S)$ being a group or having an identity.

1.4 Proposition. *If $I(S)$ has an identity t_l, then for every $a, x\in S$, $atx= =tax=ax$ and $t^3=t^2$ in particular. t_l is a right identity of $I(S)$ iff t is a left identity of S provided $S=S^2$ or S is a right cancellative semigroup.*

PROOF. The first part is easy to verify. It is also easy to show that if t is a left identity of S, then t_l is a right identity of $I(S)$. Now suppose t_l is a right identity. Then $x(a_l o t_l)=xa_l$ for every $x, a\in S$ and so $tax=ax$. Thus t is a left identity of S^2. If $S=S^2$, clearly t is a left identity of S. If S is right cancellative, then $tax=ax$ implies $ta=t$ and hence t is a left identity of S.

1.5 Theorem. *Let S be a semigroup. Then $I(S)$ is a group under composition with the identity map "i" as an identity iff S is an union of groups in which every idempotent is a left identity.*

PROOF. If $I(S)$ is a group, then for any $a, x\in S$, there exists b such that $x(a_l o b_l)=xi=x(b_l o a_l)$ and so

$$(*) \qquad bax = x = abx.$$

By setting $x=a$ in $(*)$ we have $ba^2=aba$. Thus S is left regular and regular. Hence by theorem 4.3 of [1], S is an union of groups. Now if e and f are idempotents in S, then by setting $x=f$ and $a=e$ in $(*)$ we have $f=ebf$ and so $f=e(ebf)=ef$. If $x\in G_e$, the group with e as an identity, then evidently $ex=x$. If $x\notin G_e$, then $x\in G_f$, a group with an identity f and $e\neq f$. Since $ef=f$, we have $efx=fx$ and $ex=x$. Thus every idempotent is a left identity. Conversely, let S be an union of groups such that every idempotent is a left identity. Let a_l be a left translation and $x\in S$. Then $x\in G_f$, a group with an identity f. Consider $a\in G_e$. Then there exists b such that $ba=e=ab$ and $abx=bax=ex=x$. Thus $x(b_l\circ a_l)=x(a_l\circ b_l)=x$ and hence $I(S)$ is a group.

In 1.5 the condition that every idempotent is a left identity is necessary. The semigroup $S=\{a, b, ab\colon ab=ba; a^2=a; b^2=b\}$ is an union of groups. But the set of all inner left translations is not a group since for $x=f$ and $a=e$, $(*)$ is not solvable.

2. *Vector maps.*

By 1.1 semigroups having no idempotents do not possess vector maps. Furthermore we have

2.1 Theorem. *Let S be a semigroup with $|S|>1$. Then S has no vector maps if any one of the following conditions is satisfied:*

i) *S has no idempotents except identity.*

ii) *S is a group.*

iii) *S is a left cancellative semigroup with unique idempotent.*

iv) *S is a commutative cancellative semigroup.*

PROOF. Suppose S has a vector map F.

To prove (i): If $x\in S$ and $xF=y$, then $y^3=y^4$ by 1.1. But by hypothesis we must have $y^3=1$. Then $y=1\cdot y=y^4=1$. Thus $xF=1$ for every $x\in S$ and so $1=x^2F=(xF)x=x$. This implies $|S|=1$, which is a contradiction.

To prove (iii): If $x\in S$ and $xF=y$, then $y^3=y^4$, which implies $y=y^2$ by cancellative condition. Since S has a unique idempotent, say $y, xF=y$ for every $x\in S$. Now for $t\in S, (yF)t=(yt)F$. Hence $yt=y$. Since idempotents are left identities in left cancellative semigroups $yt=t$. Thus $t=y$ and so $|S|=1$. (ii) and (iv) are now evident.

2.2 Theorem. *Let S be a semigroup containing an unique idempotent e. Then S has vector maps iff e is the zero of S.*

PROOF. Let F be a vector map on S. Suppose $x\in S$ and $xF=y$. Then by 1.1, $y^3=y^4$ and so $y^3=e$. Then $ey=y^3\cdot y=y^3=e$. Since $(yx)F=(yF)x=y^2x$, as before we must have $ey^2x=e$. This implies $ex=e$ and so xe is an idempotent, which is e itself. Thus e is a zero of S. If e is a zero of S, then the map carrying every element into e is a vector map.

If a semigroup has more than one idempotent, then 2.2 need not be true as can be seen in an idempotent semigroup without 0, which has the identity map as a vector map.

3. *Right or left zero semigroups.*

3.1 Lemma. *If T is a vector map on a left cancellative semigroup, then the image of every element under T is an idempotent and the image of an idempotent is itself.*

PROOF. If $aT=b$, then $b^3=b^4$ by 1.1 and so $b=b^2$ by left cancellative condition. If e is an idempotent and if $eT=l$, then l is an idempotent from the above. Since $eT=(eT)e$; $l=le$. Then $l=e$ by left cancellative condition.

3.2 Theorem. *For a semigroup S the following are equivalent:*

i) *S is a right zero semigroup.*

ii) *S is a right group with vector maps.*

iii) *S is a left cancellative semigroup with identity map as a vector map.*

iv) *S is an idempotent semigroup in which the set of vector maps form a group under composition with identity map "i" as an identity such that every vector map is its own inverse.*

v) *S is a left cancellative semigroup in which every inner left translation is a vector map. Furthermore the right zero semigroup has only one vector map, which is the identity map.*

PROOF. (ii)$\Rightarrow$(i): Let $a\in S$. Since vector maps exist by hypothesis, S has idempotents by 1.1. If e is an idempotent, then e is a left identity since S is a left cancellative semigroup. Then $a=ea$. If T is a vector map, then $aT=(eT)a=ea$ by 3.1 and hence $aT=a$. Thus the only vector map is an identity map. Hence by 1.2 S is an idempotent semigroup. Thus (i) is evident.

(iii)$\Rightarrow$ (i): By 1.2 since identity map is a vector map, S is an idempotent semigroup. Hence follows from the definition.

(iv) $\Rightarrow$ (i): Let $x\in S$. Then for every $a\in S, x(a_l\circ a_l)=xi$ and so $ax=a$. Thus every element is a left identity. Since S is an idempotent semigroup, the conclusion follows.

(v) $\Rightarrow$ (i): Let $x\in S$. Since x_l is a vector map by hypothesis, $a(x_l\circ x_l)=(ax_l)^2$ for every $a\in S$ and so $x^2a=(xa)^2$. By setting $a=x$, we have $x^3=x^4$. Then by left cancellative condition $x=x^2$. Thus S is an idempotent semigroup. Hence S is a right zero semigroup.

The converse and the latter part are evident from the above.

In conclusion we shall provide another characterization of right zero semigroups, which can be patterned similar to the following result. For this we need.

Definition. A semigroup S is said to be V-irreducible iff S has vector maps and $xV=S$ for every $x\in S$, where V is the set of all vector maps on S.

3.3 Theorem. *A semigroup S is a left zero semigroup iff S is V-irreducible.*

PROOF. Let S be V-irreducible. If $x\in S$, then there exists a vector map F such that $xF=x$. Since $x(F\circ F)=(xF)^2$, we must have $x=x^2$. Thus S is an idempotent semigroup. Let $a\in S$. Then for any $b\in S$, there exists an $G\in V$ such that $b=aG$. So $b=aG=(aG)a$. Thus $S=Sa$. Hence S is left simple. Combining the facts S is left simple and S is an idempotent semigroup, we have that S is a left zero semigroup. Conversely if S is a left zero semigroup, then for any $b\in S$, the map carrying every element into b is a vector map. Thus $aV=S$ for $a\in S$.

References

[1] A. H. CLIFFORD and G. B. PRESTON, The algebraic theory of semigroups, Vol. 1, Survey 7, *Amer. Math. Soc.* (1961).

[2] T. TAMURA, Notes on translations of a semigroup, *Kodai Math. Sem. Rep.*, **1** (1958), 9—26 (MR20, 1066).

BOWLING GREEN STATE UNIVERSITY,
OHIO

(Received September 19, 1974.)