Absolute total-effective Nörlund method

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1. *Introduction and the main result*

Let $\{p_n\}$ be a sequence of real numbers such that P_n^0 or $P_n = \sum_{k=0}^{n} p_k \neq 0$ and $P_n^1 = \sum_{k=0}^{n} P_k \neq 0$. The Nörlund method (N, p_n) is associated with the series to sequence transformation which transforms a series $\Sigma\, a_n$ to the sequence $\{t_n^p(a)\}$ defined by

$$t_n^p(a) = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} a_k.$$

A sequence $\{s_n\} \in BV$, *if* $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$; *and* $\{s_n\} \in B$, *if* $\{s_n\}$ *is a bounded sequence.*

The object of the present paper is to improve the theorem proved in [3] and we shall throughout use the definitions and notations contained therein. We shall prove the following.

Theorem. *If the sequence* $\{p_n\}$ *is such that*

$$(1.1) \qquad \{R_n\} \equiv \{(n+1) p_n / P_n\} \in BV \quad \text{and} \quad S_n^* = \left\{ \frac{1}{P_n} \sum_{k=0}^{n} \frac{|P_k|}{k+1} \right\} \in B,$$

then the (N, P_n) *method is absolute total-effective.*

Writing $P_n^{1*} = \sum_{k=0}^{n} |P_k|$, P_n^* or $P_n^{0*} = \sum_{k=0}^{n} |p_k|$ and $\{R_n^1\} \equiv \{(n+1) P_n / P_n^1\}$, the above result under the additional conditions viz.,

$$(1.2) \qquad P_n^{j*} = O(|P_n^j|), \quad j = 0, 1 \quad \text{and} \quad \{R_n^1\} \in BV,$$

was proved in [3]. From the theorem proved here we shall deduce in the last section the following which includes *inter alia* some earlier results and provides a shorter alternative proof of them (see e.g. [2]).

Corollary. *The* $(C, 1)(N, p_n)$ *method is absolute total-effective, if* $P_n > 0$ *and* (1.1) *holds.*

2. *Some preliminary results*

Lemma 1. *Let* $\{a_k\}$ *be a given sequence, then for any* x, *we have*

$$(1-x)\sum_{k=m}^{n} a_k x^k = -\sum_{k=m}^{n} \Delta a_{k-1} x^k + a_{m-1}x^m - a_n x^{n+1},$$

where m, n *are integers such that* $0 \leqq m \leqq n$ *and* $\Delta a_k = a_k - a_{k+1}$.

The proof of Lemma 1 is direct.

Lemma 2. *If* $\{R_n\} \in B$ *and* $\{S_n^*\} \in B$, *then* $P_n^* \leqq K|P_n|$, $\{1/R_n^1\} \in B$,

$$|P_k| \sum_{n=k}^{\infty} \frac{1}{(n+1)|P_n|} \leqq K(k = 0, 1, 2, \ldots), \tag{2.1}$$

and $|P_n| \to \infty$, *as* $n \to \infty$.

Proof. That $P_n^* \leqq K|P_n|$ follows directly from the hypotheses, when we observe that

$$P_n^* = \sum_{k=0}^{n} |R_k| \frac{|P_k|}{k+1} \leqq K \sum_{k=0}^{n} \frac{|P_k|}{k+1} \leqq K|P_n|.$$

Using this result and observing that

$$|P_n^1| \leqq \sum_{k=0}^{n} P_k^* \leqq (n+1)P_n^* \leqq K(n+1)|P_n|,$$

we prove that $\{1/R_n^1\} \in B$. For the proof of (2.1) reference may be made to ([4], pp. 13—14). Finally, since $P_n^* \leqq K|P_n|$, the convergence of the series in (2.1) leads to the result that $|P_n| \to \infty$, as $n \to \infty$.

Lemma 3. *If* $p_n = o(P_n)$ *and* $|P_n| \to \infty$, *as* $n \to \infty$, *then* $P_n^{1*} \leqq K|P_n^1|$ *for all* n.

Proof. Since $p_n = o(P_n)$, for a given positive $\varepsilon < 1$, we have a positive integer N, such that $-\varepsilon < p_n/P_n < \varepsilon$, whenever $n > N$. Writing $P_{n-1}/P_n = 1 - p_n/P_n$, we observe that for $n > N$, $0 < 1 - \varepsilon < P_{n-1}/P_n < 1 + \varepsilon$ (cf. proof of Lemma 2 in [6]) and this implies that $P_N, P_{N+1}, \ldots,$ are all positive or all negative. Since N is fixed and $P_n^1 \neq 0$, we have $P_k^{1*} \leqq K|P_k^1|$, for $0 \leqq k \leqq N$. If $k > N$, $P_k^{1*} = P_N^{1*} + \delta(P_k^1 - P_N^1)$, where $\delta = -1$ or $+1$, according as $P_{N+1}, P_{N+2}, \ldots,$ are all negative or all positive. Thus, $P_k^{1*} < K + K|P_k^1|$ and the result of the lemma follows, when we observe that $P_n^1 \neq 0$ for all n, and the sequence $\{P_n\}$ is ultimately a positive sequence or a negative sequence and by hypothesis $|P_n| \to \infty$ as $n \to \infty$.

Lemma 4. *If* $P_n^{1*} \leqq K|P_n^1|$ *for all* n, *and* $\{R_n\} \in B$, *then* $\{R_n^1\} \in B$. *Further* (2.1) *implies the following, whenever* $\{R_n^1\} \in B$,

$$|P_k| \sum_{n=k}^{\infty} \frac{1}{|P_n^1|} \leqq K(k = 0, 1, 2, \ldots). \tag{2.2}$$

Proof. We write

$$(n+1)P_n = -\sum_{k=0}^{n} \Delta\{kP_{k-1}\} + P_0 = \sum_{k=0}^{n} P_{k-1} + \sum_{k=0}^{n} R_k P_k + P_0$$

and, therefore, since $\{R_n\}\in B$,

$$(n+1)\,|P_n| \leqq K\,P_n^{1*} \leqq K\,|P_n^1|,$$

by virtue of the hypothesis that $P_n^{1*}\leqq K|P_n^1|$. Thus $\{R_n^1\}\in B$. The remaining part of the lemma follows, when we write

$$\sum_{n=k}^{\infty}\frac{1}{|P_n^1|}=\sum_{n=k}^{\infty}\frac{|R_n^1|}{(n+1)\,|P_n^1|}.$$

Lemma 5. *If* $P_n^*\leqq K|P_n|$ *for all* n, *then uniformly in* $0<t\leqq\pi$

$$\left|\sum_{k=0}^{n}P_k\exp ikt\right|\leqq Kt^{-1}\,|P_n|.$$

Lemma 4 is essentially the same as Lemma 3 in [4].

Lemma 6. *Let* $q_n>0,\ Q_n=\sum_{k=0}^{n}q_k$ *and* $d_n=\sum_{k=0}^{n}q_kc_k/Q_n$, *then if the sequence* $\{c_n\}\in BV$, *the sequence* $\{d_n\}\in BV$.

Lemma 6 is due to MOHANTY ([5], Lemma 4).

Lemma 7. *If* $\{R_n\}\in B$ *and* $P_n^{1*}\leqq K|P_n^1|$, *then* $|N,p_n|\subset|N,P_n|$, *i.e. every series summable* $|N,p_n|$ *is also summable* $|N,P_n|$.

PROOF. We first observe that $\{1/P_n^1\}\in BV$. For,

$$\sum_{n=1}^{\infty}\left|\frac{1}{P_{n-1}^1}-\frac{1}{P_n^1}\right|\leqq K\sum_{n=1}^{\infty}\frac{|P_n|}{P_n^{1*}P_{n-1}^{*}}=K\sum_{n=1}^{\infty}\left(\frac{1}{P_{n-1}^{1*}}-\frac{1}{P_n^{1*}}\right)\leqq K,$$

since $P_n^{1*}\leqq K|P_n^1|$ and $\{P_n^{1*}\}$ is positive and monotonic increasing.

Since the hypothesis $\{R_n\}\in B$, implies that $p_n=o(P_n)$, $n\to\infty$, it follows from the proof of Lemma 3 that there exists a positive integer N such that $P_N, P_{N+1}, \ldots,$ are all positive or all negative. We now write n-th (N, P_n) mean of $\Sigma\, a_n$ as (cf. [1], p. 360)

$$t_{n+N}^P(a)=\frac{1}{P_{n+N}^1}\sum_{k=0}^{N-1}P_kt_k^P(a)+\frac{\sum_{k=0}^{n}P_{N+k}t_{N+k}^P(a)}{P_{N+n}^1-P_{N-1}^1}\left(1-\frac{P_{N-1}^1}{P_{N+n}^1}\right)=$$

$$=\frac{K}{P_{n+N}^1}+\mu_n(a)\left(1-\frac{P_{N-1}^1}{P_{n+N}^1}\right)$$

say. Since $\{1/P_n^1\}\in BV$, in order to prove that $\{t_n^P(a)\}\in BV$, it is sufficient to show that $\{\mu_n(a)\}\in BV$. However, if $\Sigma\, a_n$ is summable $|N,p_n|$, then $\{t_n^P(a)\}\in BV$ and by Lemma 6, $\{\mu_n(a)\}\in BV$, since $P_N, P_{N+1}, \ldots,$ are all positive or all negative.

Lemma 8. *If* $\theta(t)\in BV(0, \pi)$ *and* (1.1) *holds then the series*

$$\sum_n \int_0^\pi \Theta(t) \cos nt\, dt$$

is summable $|N, p_n|$.

Lemma 8 follows from the proof of Theorem C in [4], when we appeal to the Lemma 2 of the present paper.

3. *Proof of the Theorem*

Integrating by parts, we have for the Fourier series $L(x)$ (cf. [3])

$$\frac{\pi}{2} a_k = \int_0^\pi \varphi(t) \cos kt\, dt = \int_0^\pi \{t \cos kt - \int_0^t \cos ku\, du\} d\varphi_1(t) =$$

$$= \int_0^\pi t \cos kt\, d\varphi_1(t) + \int_0^\pi \varphi_1(t) \cos kt\, dt = u_k + v_k,$$

say. Since x is $|F_1|$-regular, $\varphi_1(t)\in BV(0, \pi)$ and $\sum_n v_n$ is summable $|N, p_n|$ by Lemma 8. The $|N, P_n|$ summability of $\sum_n v_n$ now follows, when we appeal to Lemma 7. Considering the series $\sum_n u_n$, we observe that (cf. [3])

$$t_n^P(u) - t_{n-1}^P(u) = \int_0^\pi \left\{\frac{t}{P_n^1 P_{n-1}^1} \sum_{k=0}^n (P_n^1 P_k - P_k^1 P_n) \cos (n-k) t\right\} d\varphi_1(t).$$

Since $\int_0^\pi |d\, \varphi_1(t)| \leqq K$, in order to show that $\{t_n^P(u)\}\in BV$, it is sufficient to demonstrate that uniformly in $0 < t \leqq \pi$

$$\Sigma = t \sum_{n=1}^\infty \left| \frac{1}{P_n^1 P_{n-1}^1} \sum_{k=0}^n (P_n^1 P_k - P_k^1 P_n) \exp ikt \right| \leqq K. \tag{3.1}$$

By virtue of Lemma 1, we have

$$\sum_{k=0}^n (P_n^1 P_k - P_n P_k^1) \exp ikt = (1 - \exp it)^{-1} \sum_{k=0}^n (P_n^1 P_k - P_n p_k) \exp ikt.$$

Thus,

$$\begin{aligned} \Sigma \leqq\ & t \sum_{n=1}^\tau \left| \frac{1}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^n |P_n^1 P_k - P_n P_k^1| + \\ & + K \sum_{n=\tau+1}^\infty \left| \frac{1}{P_n P_{n-1}} \right| \sum_{k=0}^{\tau-1} |P_n^1 p_k - P_n P_k| + \\ & + K \sum_{n=\tau+1}^\infty \left| \frac{1}{P_{n-1}^1} \sum_{k=\tau}^n \left(\frac{p_k}{P_k} - \frac{P_n}{P_n^1}\right) P_k \exp ikt \right| = \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned} \tag{3.2}$$

say. Since by Lemma 3, $P_n^{1*} \leqq K|P_n^1|$, we have

$$\Sigma_1 \leqq t \sum_{n=1}^{\tau} \frac{1}{|P_{n-1}^1|} \sum_{k=0}^{n} |P_k| + t \sum_{n=1}^{\tau} \left| \frac{P_n}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^{n-1} P_k^{1*} \leqq \tag{3.3}$$

$$\leqq Kt \sum_{n=1}^{\tau} \left| \frac{P_n^1}{P_{n-1}^1} \right| + Kt \sum_{n=1}^{\tau} |R_n^1| \leqq K.$$

by virtue of Lemmas 2—4.

Again using the result that $P_n^{1*} \leqq K|P_n^1|$, we have from Lemma 4,

$$\Sigma_2 \leqq K|P_{\tau-1}| \sum_{n=\tau+1}^{\infty} \frac{1}{|P_{n-1}^1|} + K \sum_{n=\tau+1}^{\infty} \left| \frac{P_n}{P_n^1 P_{n-1}^1} \right| \sum_{k=0}^{\tau-1} P_k^* \leqq \tag{3.4}$$

$$\leqq K + K\tau P_{\tau-1}^* \sum_{n=\tau+1}^{\infty} \frac{|R_n^1|}{n|P_{n-1}^1|} \leqq K,$$

since $P_n^* \leqq K|P_n|$, by virtue of Lemma 2.

Applying Abel's transformation, we write

$$\Sigma_3 \leqq K \sum_{n=\tau+1}^{\infty} \left| \frac{1}{P_{n-1}^1} \sum_{k=\tau}^{n-1} \Delta\left(\frac{R_k}{k+1}\right) \sum_{v=\tau}^{k} P_v \exp ivt \right| +$$

$$+ K \sum_{n=\tau+1}^{\infty} \left| \frac{1}{P_{n-1}^1} \left(\frac{p_n}{P_n} - \frac{P_n}{P_n^1} \right) \sum_{v=\tau}^{n} P_v \exp ivt \right| \leqq$$

$$\leqq K\tau \sum_{n=\tau+1}^{\infty} \frac{1}{|P_{n-1}^1|} \sum_{k=\tau}^{n-1} \left\{ \frac{|\Delta R_k|}{k+1} + \frac{|R_{k+1}|}{k(k+1)} \right\} |P_k| +$$

$$+ K\tau \sum_{n=\tau+1}^{\infty} \frac{|R_n R_n^1|}{(n+1)^2} + K\tau \sum_{n=\tau+1}^{\infty} \frac{|R_n^1 R_{n-1}^1|}{(n+1)^2} \leqq \tag{3.5}$$

$$\leqq K\tau \sum_{k=\tau}^{\infty} \frac{|\Delta R_k|}{k} |P_k| \sum_{n=k+1}^{\infty} \frac{1}{|P_{n-1}^1|} +$$

$$+ K\tau \sum_{k=\tau}^{\infty} \frac{1}{k(k+1)} |P_k| \sum_{n=k+1}^{\infty} \frac{1}{|P_{n-1}^1|} + K \leqq$$

$$\leqq K \sum_{k=\tau}^{\infty} |\Delta R_k| + K \leqq K,$$

by virtue of Lemmas 2—5 and the hypothesis that $\{R_n\} \in BV$, which implies that $P_n/P_{n-1} \to 1$, as $n \to \infty$.

Combining (3.2)-(3.5), we prove (3.1) and thus $\sum_n u_n$ is summable $|N, P_n|$.

This completes the proof of the $|F_1|$-effectiveness part of the theorem.

The rest of the theorem follows from the foregoing proof of the $|F_1|$-effectiveness, when we refer to section 5 of [3].

4. *Proof of the Corollary*

We need he following additional lemma.

Lemma 9. *If $P_n > 0$, $P_n \to \infty$, $\{R_n^1\} \in B$ and $\{1/R_n^1\} \in B$, then the $|N, P_n|$ method is equivalent to the $|(C, 1)(N, p_n)|$ method.*

Lemma 9 follows directly from Theorems 1 and 2 of [1]. Now we see that if $P_n > 0$ and (1.1) holds then by virtue of Lemmas 2 and 4 of the present paper, the hypotheses of Lemma 9 are satisfied and in conclusion we obtain the result of the Corollary from the theorem already proved.

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