On a continuous one parameter group of operator transformations on the field of Mikusiński operators

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Introduction

Denote by C the set of complex-valued continuous functions defined on the interval $[0, \infty)$, by M the operator field of Mikusiński, and by K the field of complex numbers embedded into M. Throughout the paper convergence of operator sequences will mean type II. convergence in the sense of Mikusiński, i.e. we shall say that the sequence of operators $\{a_n\}_{n=1}^{\infty}$ converges to the operator $a \in M$, denoted by $\lim_{n\to\infty} a_n = a$, if there exists a sequence $\{q_n\}_{n=1}^{\infty}$ of functions in C, converging to the function $q \in C$, $q \neq 0$ almost uniformly on $[0, \infty)$, so that the relations $q_n a_n \in C$ $n=1, 2, \ldots$ and $qa \in C$ hold, and the sequence $\{q_n a_n\}_{n=1}^{\infty}$ converges almost uniformly to qa.

Let R_+ denote the set of positive reals. Let $\lambda \in R_+$ be arbitrary, and let us denote the operator transformation $U_\lambda \colon M \to M$ as follows (cf. [5]):

$$U_\lambda(f) \overset{\mathrm{df}}{=} \{\lambda f(\lambda t)\} \quad \text{for} \quad f = \{f(t)\} \in C,$$

and

$$U_\lambda(a) \overset{\mathrm{df}}{=} \frac{U_\lambda(f)}{U_\lambda(g)} \quad \text{for} \quad a = \frac{f}{g} \in M;\ f, g \in C, \quad g \neq 0$$

For any $\lambda \in R_+$, the transformation U_λ is a one-to-one mapping of M onto itself, operation-preserving for both operations on M. It does always map C onton itself, the image of each numerical operator is itself, and for any $\mu, \nu \in R_+$ and any operator $a \in M$ one has

$$U_\mu[U_\nu(a)] = U_{\mu\nu}(a) \tag{0.1}$$

i.e.

$$U_{\mu\nu} = U_\mu U_\nu \quad \text{for} \quad \mu, \nu > 0$$

It is easy to check that the set $\{U_\lambda\}_{\lambda>0}$ forms a commutativegroup with the unit element U_1.

In this paper we expose some properties of this one parameter group. It is well known ([2]) that the operator transformation U_λ is continuous for every fixed

$\lambda > 0$. In the first part we show that $\lim_{n\to\infty} \lambda_n = \lambda \in R_+$ and $\lim_{n\to\infty} a_n = a \in M$ together imply $\lim_{n\to\infty} U_{\lambda_n}(a_n) = U_\lambda(a)$. This result contains as particular cases both the continuity of U_λ (for $\lambda_n = \lambda$) and the continuity of the one parameter group $\{U_\lambda\}_{\lambda>0}$ (for $a_n = a$). Thereupon we prove that if the sequence of parameters $\{\lambda_n\}_{n=1}^\infty$ is supposed only to have a positive upper bound and a positive lower bound, whereas the sequence of operators $\{a_n\}_{n=1}^\infty$ also obeys the additional requirement of having a numerical operator as its limit, then the sequence $\{U_{\lambda_n}(a_n)\}_{n=1}^\infty$ will also be convergent, and its limit will be the same as that of $\{a_n\}_{n=1}^\infty$.

E. GESZTELYI has shown in [2], that if the operator $a \in M$ satisfies the equality $U_n(a) = a$ for any positive integer n, then a is a numerical operator. In the second part of the paper, we answer the question concerning the set of those parameters $\lambda \in R_+$ for wich the equality $U_\lambda(a) = a$ holds. We show that these sets are R_+ or else cyclic subgroups of R_+, considered as multiplicative group.

In the third section we investigate the limits of sequences of type $\{U_{\lambda_n}(a)\}_{n=1}^\infty$ $(a \in M)$, where $\{\lambda_n\}_{n=1}^\infty$ is a sequence of positive numbers tending to infinity. By an immediate consequence following from the theorem of E. Gesztelyi just mentioned, a convergent sequence of operators of the form $\{U_n(a)\}_{n=1}^\infty$ $(a \in M)$ can have only a number as its limit. Here we answer the question, whether such a sequence can have a limit which is no numerical operator, and how does this limit depend on the sequence of parameters.

We show that if the sequence of parameters is sufficiently "thin", then the limit is not necessarily a numerical operator. Moreover, if the sequence of parameters is e.g. $\{n!\}_{n=1}^\infty$, then any operator can be represented as the limit of a sequence of the form $\{U_{n!}(a)\}_{n=1}^\infty$, and a can be chosen also from C. From this the interesting fact follows, that there exists a subring $\hat{C}$ of C, and a congruence η on this subring, so that the factor ring $\hat{C} \bmod \eta$ is already isomorphic with M.

Finally in the last chapter we define for operators, considered as generalized functions, their integrals over $(-\infty, \infty)$: by the l-integral of a given operator $a \in M$, where $l \overset{\text{df}}{=} \{\lambda_n\}_{n=1}^\infty$ is a sequence of parameters tending to infinity, we mean the limit of the sequence of operators $\{U_{\lambda_n}(a)\}_{n=1}^\infty$, provided it exists and in a number. In this case we call the operator $a \in M$ l-integrable, and if a is l-integrable for any parameter sequence l tending to infinity, then we say it to be integrable. Finally we give a criterion for integrability being implied by l-integrability, and as an immediate consequence we obtain the equivalence of our notion of integrability with the one defined in [4].

§ 1.

Theorem 1.1. *If $\{\lambda_n\}_{n=1}^\infty$ is a sequence of positive numbers converging to a positive number λ, and $\{a_n\}_{n=1}^\infty$ is an operator sequence converging to the operator $a \in M$, then the operator sequence $\{U_{\lambda_n}(a_n)\}_{n=1}^\infty$ is also convergent and*

$$\lim_{n\to\infty} U_{\lambda_n}(a_n) = U_\lambda(a) \tag{1.1}$$

holds.

PROOF. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers converging to a positive number λ, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from C, converging almost uniformly to an $f \in C$. We are going to show that in this case

$$U_{\lambda_n}(f_n) \rightrightarrows U_\lambda(f) \tag{1.2}$$

if $n \to \infty$.

Let $T>0$ and $\varepsilon>0$ be arbitrary fixed numbers. Let R be an upper bound of the sequence $\{\lambda_n\}_{n=1}^{\infty}$, and let $K>0$ be so that

$$|f(t)| < K \tag{1.3}$$

for any $0 \leqq t \leqq RT$. Also, let $\delta>0$ be so that for any $t_1, t_2 \in [0, RT]$ satisfying $|t_1-t_2|<\delta$ the inequality

$$|f(t_1)-f(t_2)| < \frac{\varepsilon}{3\lambda} \tag{1.4}$$

holds. Let N be so that $n>N$ implies the inequalities

$$|f_n(t)-f(t)| < \frac{\varepsilon}{3R} \quad \text{for} \quad 0 \leqq t \leqq RT \tag{1.5}$$

$$|\lambda_n - \lambda| < \frac{\varepsilon}{3K}$$

and

$$|\lambda_n - \lambda| < \frac{\delta}{T}$$

Since for any $0 \leqq t \leqq T$ and any natural number n one has $0 \leqq \lambda_n t \leqq RT$, $0 \leqq \lambda t \leqq RT$ too holds, and (1.3), (1.4) and (1.5) together imply that for $n>N$ and any $t \in [0, T]$ the inequality

$$|\lambda_n f_n(\lambda_n t) - \lambda f(\lambda t)| \leqq |\lambda_n f_n(\lambda_n t) - \lambda_n f(\lambda_n t)| + |\lambda_n f(\lambda_n t) - \lambda f(\lambda_n t)| +$$
$$+ |\lambda f(\lambda_n t) - \lambda f(\lambda t)| = \lambda_n |f_n(\lambda_n t) - f(\lambda_n t)| + |\lambda_n - \lambda| |f(\lambda_n t)| +$$
$$+ \lambda |f(\lambda_n t) - f(\lambda t)| < R\frac{\varepsilon}{3R} + \frac{\varepsilon}{3K}K + \lambda\frac{\varepsilon}{3\lambda} = \varepsilon$$

holds.

Since T and ε have been arbitrary, (1.2) does in fact hold.

Let now $\{a_n\}_{n=1}^{\infty}$ be an arbitrary convergent operator sequence, and let

$$a = \lim_{n\to\infty} a_n.$$

Then there exist a sequence $\{q_n\}_{n=1}^{\infty}$ of functions from C, and a function $q \in C$, $q \neq 0$ so that

$$q_n \rightrightarrows q \quad (n \to \infty)$$

$$q_n a_n \in C \quad (n = 1, 2, \ldots), \quad qa \in C$$

and

$$q_n a_n \rightrightarrows qa \quad (n \to \infty)$$

Let us now apply (1.2) to the sequences $\{q_n\}_{n=1}^{\infty}$ and $\{q_n a_n\}_{n=1}^{\infty}$. We obtain

$$U_{\lambda_n}(q_n) \rightrightarrows U_\lambda(q) \quad (n \to \infty)$$
$$U_{\lambda_n}(q_n a_n) \rightrightarrows U_\lambda(qa) \quad (n \to \infty)$$

and this is equivalent with (1.1).

Theorem 1.2. *If* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of operators converging to a numerical operator* $\alpha \in K$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *is a sequence of positive numbers having an upper and a positive lower bound, then the operator sequence* $\{U_{\lambda_n}(a_n)\}_{n=1}^{\infty}$ *is convergent, and*

$$\lim_{n\to\infty} U_{\lambda_n}(a_n) = \alpha \tag{1.6}$$

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an operator sequence converging to a numerival operator $\alpha \in K$ and let $\{q_n\}_{n=1}^{\infty}$ be that sequence of functions from C for which the following relations hold:

$$\begin{gathered} q_n \rightrightarrows q \quad (n \to \infty), \quad q \in C, \quad q \neq 0 \\ p_n \overset{\text{df}}{=} a_n q_n \in C \quad (n = 1, 2, \ldots) \\ p_n \rightrightarrows \alpha q \quad (n \to \infty) \end{gathered} \tag{1.7}$$

Without loss of generality we can suppose that there exists a function $g \in C$ so that

$$q = \{1\} g \tag{1.8}$$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers having an upper and a positive lower bound. Let $0 < \lambda' < \lambda''$ be those two numbers for which

$$\lambda' < \lambda_n < \lambda'' \tag{1.9}$$

holds for all natural number n and let

$$\lambda \overset{\text{df}}{=} \frac{\lambda'}{\lambda''} \tag{1.10}$$

First we show that for any numbers $T > 0$ and $\varepsilon > 0$ there exists a continuous function f defined on $[\lambda, 1] \times [0, \infty)$ and satisfying the inequality

$$\left| \int_0^t q(\tau) f(\mu, t-\tau)\, d\tau - \mu q(\mu t) \right| < \varepsilon \tag{1.11}$$

for all $\mu \in [\lambda, 1]$ and $t \in [0, T]$.

Let $T > 0$ and $\varepsilon > 0$ be arbitrary fixed numbers. Let moreover $\varrho = \Lambda(g)$ (see [2]). Now the function $e^{\varrho s} g = \{g(t+\varrho)\}$ does not identically vanish in any right neighborhood of zero, and so by the theorem of Foias, (see [1]) for every $\mu \in [\lambda, 1]$

there exists a $k_\mu \in C$ such that

$$\int_0^{T-\varrho} \left| \int_0^s g(\sigma+\varrho)\, k_\mu(s-\sigma)\, d\sigma - \mu^2 g\big(\mu(s+\varrho)\big) \right| ds < \frac{\varepsilon}{2}$$

hence by the substitutions $s+\varrho=t$ and $\sigma+\varrho=\tau$ we get

$$\int_\varrho^T \left| \int_\varrho^t g(\tau)\, k_\mu(t-\tau)\, d\tau - \mu^2 g(\mu t) \right| dt < \frac{\varepsilon}{2}$$

and from this, taking into account the fact that if $\varrho>0$ then $g(t)=0$ for all $0 \leqq t \leqq \varrho$ and thus $\mu^2 g(\mu t)=0$ since $\mu \leqq 1$, it follows that

$$\int_0^T \left| \int_0^t g(\tau)\, k_\mu(t-\tau)\, d\tau - \mu^2 g(\mu t) \right| dt < \frac{\varepsilon}{2}$$

By interchanging here integration and the taking of absolute value, and by making use of (1.8) we obtain that the inequality

$$\left| \int_0^t q(\tau)\, k_\mu(t-\tau)\, d\tau - \mu q(\mu t) \right| < \frac{\varepsilon}{2} \tag{1.12}$$

is valid for all $0 \leqq t \leqq T$.

On the other hand $\{\mu q(\mu t)\}$ considered as a function of two variables is continuous and therefore uniformly continuous on the closed rectangle $[\lambda, 1] \times [0, \infty)$ and this implies the existence of a positive integer m such that $t \in [0, T]$, $\mu', \mu'' \in [\lambda, 1]$ and $|\mu'-\mu''| < (1-\lambda)/m$ together imply the inequality

$$|\mu' q(\mu' t) - \mu'' q(\mu'' t)| < \frac{\varepsilon}{2} \tag{1.13}$$

Let

$$\mu_i \overset{\text{df}}{=} \frac{i+(m-i)\lambda}{m} \quad (i = 0, 1, 2, \dots m) \tag{1.14}$$

and let us define the function f on the rectangle $[\lambda, 1] \times [0, \infty)$ as follows: let

$$f(\mu, t) \overset{\text{df}}{=} \frac{\mu_i - \mu}{\mu_i - \mu_{i-1}}\, k_{\mu_{i-1}}(t) + \frac{\mu - \mu_{i-1}}{\mu_i - \mu_{i-1}}\, k_{\mu_i}(t) \tag{1.15}$$

if $0 \leqq t < \infty$, $\mu_{i-1} \leqq \mu \leqq \mu_i$ and $k_{\mu_i} \in C$ $(i=0, 1, 2, \dots m)$ is a function satisfying (1.12). As an immediate consequence of (1.14), f is continuous on $[\lambda, 1] \times [0, \infty)$, moreover $\mu_{i-1} \leqq \mu \leqq \mu_i$ implies $|\mu_i - \mu| < (1-\lambda)/m$ and $|\mu - \mu_{i-1}| < (1-\lambda)/m$. From this however we infer by (1.12), (1.13) and on the basis of (1.15) that the following inequality is valid for any $\lambda \leqq \mu \leqq 1$ and $0 \leqq t \leqq T$:

$$\left|\int_0^t q(\tau) f(\mu, t-\tau)\, d\tau - \mu q(\mu t)\right| =$$

$$= \left|\frac{\mu_i - \mu}{\mu_i - \mu_{i-1}}\left[\int_0^t q(\tau)\, k_{\mu_{i-1}}(t-\tau)\, d\tau - \mu q(\mu t)\right] + \right.$$

$$\left. + \frac{\mu - \mu_{i-1}}{\mu_i - \mu_{i-1}}\left[\int_0^t q(\tau)\, k_{\mu_i}(t-\tau)\, d\tau - \mu q(\mu t)\right]\right| \leqq$$

$$\leqq \frac{\mu_i - \mu}{\mu_i - \mu_{i-1}}\left[\left|\int_0^t q(\tau)\, k_{\mu_{i-1}}(t-\tau)\, d\tau - \mu_{i-1} q(\mu_{i-1} t)\right| + \right.$$

$$\left. + |\mu_{i-1} q(\mu_{i-1} t) - \mu q(\mu t)|\right] +$$

$$+ \frac{\mu - \mu_{i-1}}{\mu_i - \mu_{i-1}}\left[\left|\int_0^t q(\tau)\, k_{\mu_i}(t-\tau)\, d\tau - \mu_i q(\mu_i t)\right| + \right.$$

$$\left. + |\mu_i q(\mu_i t) - \mu q(\mu t)|\right] < \varepsilon$$

where $\mu_{i-1} \leqq \mu \leqq \mu_i$.

Let now $\{f_k\}_{k=1}^{\infty}$ be a sequence of continuous functions defined on $[\lambda, 1] \times \times [0, \infty)$, each member of which satisfies the inequality

$$\left|\int_0^t q(\tau) f_k(\mu, t-\tau)\, d\tau - \mu q(\mu t)\right| < \frac{1}{2k\lambda''} \tag{1.16}$$

for any $\mu \in [\lambda, 1]$ and $t \in [0, k\lambda'']$.

Let

$$\delta_k \overset{\text{df}}{=} \max\{|f_k(\mu, t)| \,|\, \lambda \leqq \mu \leqq 1, \quad 0 \leqq t \leqq k\lambda''\} \quad (k = 1, 2, \ldots) \tag{1.17}$$

Then, by (1.7), for any natural number k there exists an integer N_k so, that $n \geqq N_k$ implies the validity of the inequalities

$$|p_n(t) - \alpha q(t)| < \frac{2-|\alpha|}{2(k\lambda'')^2 \delta_k} \tag{1.18}$$

$$|q_n(t) - q(t)| < \frac{1}{2(k\lambda'')^2 \delta_k}$$

for any $0 \leqq t \leqq k\lambda''$. Without restricting generality we can suppose that $N_k < N_{k+1}$ $(k = 1, 2, \ldots)$.

Let

$$g_n \overset{\text{df}}{=} f_1 \quad \text{for} \quad n < N_1$$

and

$$g_n \overset{\text{df}}{=} f_k \quad \text{for} \quad N_k \leqq n < N_{k+1}$$

From this and from (1.9), (1.16), (1.17) and (1.18) we are now able to infer that if k is any natural number, then for $n > N_k$ and $0 \leqq t \leqq k$ the following inequalities hold, where $r(r \geqq k)$ denotes the natural number satisfying $N_r \leqq n < N_{r+1}$:

$$\left|\lambda_n \int_0^{\lambda_n t} p_n(\tau) g_n\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau - \lambda' \alpha q(\lambda' t)\right| \leqq$$

$$\leqq \lambda_n \left|\int_0^{\lambda_n t} [p_n(\tau) - \alpha q(\tau)] f_r\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau\right| +$$

$$+ \lambda_n |\alpha| \left|\int_0^{\lambda_n t} q(\tau) f_r\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau - \frac{\lambda'}{\lambda_n} q\left(\frac{\lambda'}{\lambda_n} \lambda_n t\right)\right| <$$

$$< \lambda_n^2 t \frac{2 - |\alpha|}{2(r\lambda'')^2 \delta_r} \delta_r + \lambda_n |\alpha| \frac{1}{2r\lambda''} \leqq (2 - |\alpha|) \frac{k}{2r^2} + |\alpha| \frac{1}{2r} \leqq \frac{1}{k}$$

and

$$\left|\lambda_n \int_0^{\lambda_n t} q_n(\tau) g_n\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau - \lambda' q(\lambda' t)\right| \leqq$$

$$\leqq \lambda_n \left|\int_0^{\lambda_n t} [q_n(\tau) - q(\tau)] f_r\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau\right| +$$

$$+ \lambda_n \left|\int_0^{\lambda_n t} q(\tau) f_r\left(\frac{\lambda'}{\lambda_n},\ \lambda_n t - \tau\right) d\tau - \frac{\lambda'}{\lambda_n} q\left(\frac{\lambda'}{\lambda_n} \lambda_n t\right)\right| <$$

$$< \lambda_n^2 t \frac{1}{2(r\lambda'')^2 \delta_r} \delta_r + \lambda_n \frac{1}{2r\lambda''} \leqq \frac{1}{k}$$

This means that

$$U_{\lambda_n}\left(p_n g_n\left(\frac{\lambda'}{\lambda_n}\right)\right) \rightrightarrows \alpha U_{\lambda'}(q) \quad (n \to \infty)$$

and

$$U_{\lambda_n}\left(q_n g_n\left(\frac{\lambda'}{\lambda_n}\right)\right) \rightrightarrows U_{\lambda'}(q) \quad (n \to \infty)$$

but

$$U_{\lambda_n}(a_n) = U_{\lambda_n}\left(\frac{p_n}{q_n}\right) = \frac{U_{\lambda_n}\left(p_n g_n\left(\frac{\lambda'}{\lambda_n}\right)\right)}{U_{\lambda_n}\left(q_n g_n\left(\frac{\lambda'}{\lambda_n}\right)\right)} \quad (n = 1, 2, \ldots)$$

so that the operator sequence $\{U_{\lambda_n}(a_n)\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n \to \infty} U_{\lambda_n}(a_n) = \alpha.$$

§ 2.

Let $a \in M$ be arbitrary, and let us define the set $H(a)$ as follows:

$$H(a) \stackrel{\text{df}}{=} \{\lambda \mid \lambda \in R_+,\ U_\lambda(a) = a\} \tag{2.1}$$

Let us denote by $\langle \alpha \rangle$ the cyclic subgroup generated by the element α of the multiplicative group R_+, and let $\langle 1 \rangle$ be the one-element trivial subgroup of R_+.

Theorem 2.1. *For any operator $a \in M$, $H(a)$ is a trivial or a cyclic subgroup of the multiplicative group R_+, and conversely, for any trivial or cyclic subgroup H of R_+ there exists an operator $a \in M$, such that $H = H(a)$.*

PROOF. Let $a \in M$ be an arbitrary operator. First we show that $H(a)$ is a subgroup of R_+, closed in R_+ with respect to the usual topology of reals.

Clearly $1 \in H(a)$. Let now be $\mu, \nu \in H(a)$ arbitrary. Then by (0.1) the following equality holds:

$$U_{\frac{\mu}{\nu}}(a) = U_{\frac{1}{\nu}}[U_\mu(a)] = U_{\frac{1}{\nu}}(a) = U_{\frac{1}{\nu}}[U_\nu(a)] = U_1(a) = a$$

i.e. $\mu/\nu \in H(a)$, hence $H(a)$ is a subgroup of R_+.

Let now be $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of elements of $H(a)$, tending to $\lambda \in R_+$, i.e. a sequence satisfying

$$U_{\lambda_n}(a) = a \quad (n = 1, 2, \dots)$$

$$\lim_{n\to\infty} \lambda_n = \lambda \in R_+$$

From this we infer by theorem 1.1 that

$$a = \lim_{n\to\infty} U_{\lambda_n}(a) = U_\lambda(a)$$

i.e. that $\lambda \in H(a)$. Thus we have shown that $H(a)$ contains all its accumulation point belonging to R_+, i.e. that $H(a)$ is closed in R_+.

Now we show that a subgroup of R_+ closed in R_+ is either trivial or else a cyclic subgroup.

Of course, $\langle 1 \rangle$ is a closed subgroup of R_+. Let now $H \neq \langle 1 \rangle$ be an arbitrary subgroup of R_+, closed in R_+. Since H has an element different from 1, it also contains some element which is larger than 1. Let

$$\alpha \stackrel{\text{df}}{=} \inf\{\lambda \mid \lambda \in H, \lambda > 1\} \tag{2.2}$$

H being closed in R_+, one has $\alpha \in H$. Suppose $\alpha > 1$. We show that $H = \langle \alpha \rangle$. Indeed, let $\beta \in H$ arbitrary. Then there exists an integer n so that

$$\alpha^n \leqq \beta < \alpha^{n+1}$$

From this

$$1 \leqq \frac{\beta}{\alpha^n} < \alpha$$

follows, but $\beta/\alpha^n \in H$ and so by (2.2) $\beta/\alpha^n = 1$, i.e. $\beta = \alpha^n$. Thus $H = \langle\alpha\rangle$. Let now be $\alpha = 1$, and let $\beta \in R_+$ be arbitrary. We show that β is an accumulation point of H. Let $\varepsilon > 0$ be arbitrary. Since $\alpha = 1$, by (2.2) there exists $\nu \in H$, $\nu > 1$ for which $\nu - 1 < \varepsilon/\beta$, on the other hand there exists an integer n with

$$\nu^n \leqq \beta < \nu^{n+1}$$

Hence

$$0 \leqq \beta - \nu^n < \nu^n(\nu - 1) < \nu^n \frac{\varepsilon}{\beta} \leqq \varepsilon$$

But $\nu^n \in H$ and consequently β is an accumulation point of H. Since β has been an arbitrary element of R_+, $H = R_+$ follows.

Thus we have shown that $H(a)$ is either a trivial or a cyclic subgroup of R_+.

Let us now show that for any subgroup H of R_+, closed in R_+, there exists an operator $a \in M$, satisfying $H = H(a)$.

Since $U_\lambda(1) = 1$ and $U_\lambda(s) = s/\lambda$ for any $\lambda \in R_+$, where s denotes the operator of differentiation, we have $H(1) = R_+$ and $H(s) = \langle 1\rangle$. Let now $\alpha \in R_+$, $\alpha > 1$ be arbitrary, and let $H = \langle\alpha\rangle$. Let moreover φ be an arbitrary continuous function defined on the interval $[1, \alpha]$ and satistying the following conditions:

$$\varphi(1) = \varphi(\alpha) = 0 \quad \text{and} \tag{2.3}$$

$$\varphi(t) \neq 0 \quad 1 < t < \alpha$$

Let

$$f(t) \stackrel{\text{df}}{=} \begin{cases} 0 & \text{for} \quad t = 0, \quad \text{and} \\ \alpha^n \varphi(t\alpha^{-n}) & \text{for} \quad \alpha^n \leqq t < \alpha^{n+1}, n \end{cases}$$

an integer.

The function f so defined is of course continuous, and by (2.3) the equality

$$f(\lambda t) = \lambda f(t) \quad (\lambda \in R_+) \tag{2.4}$$

is satisfied for any $0 \leqq t < \infty$ if and only if $\lambda = \alpha^n$, where n is an integer.

Let

$$a \stackrel{\text{df}}{=} \frac{f}{\{t\}}$$

Then by (2.4) the equality

$$U_\lambda(a) = \frac{\{\lambda f(\lambda t)\}}{\{\lambda^2 t\}} = \frac{\left\{\frac{1}{\lambda} f(\lambda t)\right\}}{\{t\}} = \frac{\{f(t)\}}{\{t\}} = a$$

holds if and only if $\lambda = \alpha^n$, where n is an integer. From this however

$$H(a) = \langle\alpha\rangle$$

follows.

In [2] E. Gesztelyi has proved the following theorem:

If the operator $a \in M$ is so that

$$U_n(a) = a$$

holds for any natural number n, then a is a numerical operator.

As an immediate consequence of this and of our theorem 2.1. we get the following.

Theorem 2.2. *Let $a \in M$ be arbitrary. Then the equality*

$$H(a) = R_+$$

holds if and only if $a \in K$.

§ 3.

Let L denote the set of sequences of real numbers tending to infinity, i.e. let

$$L \stackrel{\text{df}}{=} \left\{ l \mid l \stackrel{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty}, \quad \lambda_n \in R_+ \ n = 1, 2, \dots; \ \lim_{n \to \infty} \lambda_n = +\infty \right\}$$

Definition 3.1. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence. The sequence $\{y_n\}_{n=1}^{\infty}$ will be called a generalized subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$, if there exists a sequence $\{k_n\}_{n=1}^{\infty}$ of natural numbers tending to infinity, such that

$$y_n = x_{k_n}$$

holds for each natural number n.

Clearly, if y is a subsequence in the usual sense of the sequence x, then y is also a generalized subsequence of x. Also, a generalized subsequence of a generalized subsequence of a sequence is again a generalized subsequence of that sequence. This notion of generalized subsequence has also the following property: if a real sequence or an operator sequence is convergent, then any of its generalized subsequences is also convergent, and to the same limit. In what follows, by a subsequence of a sequence we shall always mean a generalized subsequence.

Definition 3.2. Let $l \stackrel{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty}$ and $m \stackrel{\text{df}}{=} \{\mu_n\}_{n=1}^{\infty}$ be two arbitrary sequences of positive real numbers. By the product and by the quotient of l and of m we mean the sequences

$$l \cdot m \stackrel{\text{df}}{=} \{\lambda_n \mu_n\}_{n=1}^{\infty} \quad \text{and} \quad \frac{l}{m} \stackrel{\text{df}}{=} \left\{\frac{\lambda_n}{\mu_n}\right\}_{n=1}^{\infty}$$

respectively.

Definition 3.3. Let $l \in L$ be an arbitrary sequence.

By $Q(l)$ we mean the smallest multiplicative subgroup of R_+ closed in the sense of the usual topology of the reals, which contains all the positive numbers which are representable as the limit of the quotient of two subsequences of l.

Theorem 3.1. *For any sequence $l \stackrel{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty} \in L$, and for any operator $a \in M$, such that the operator-sequence $\{U_{\lambda_n}(a)\}_{n=1}^{\infty}$ is convergent, the relation*

$$H\left[\lim_{n \to \infty} U_{\lambda_n}(a)\right] \supseteq Q(l) \tag{3.1}$$

holds.

PROOF. Let $l \overset{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty} \in L$ be an arbitrary sequence, and $a \in M$ an operator, such that the sequence $\{U_{\lambda_n}(a)\}_{n=1}^{\infty}$ converges. Let us denote the limit of this operator sequence by b, i.e. let

$$b \overset{\text{df}}{=} \lim_{n \to \infty} U_{\lambda_n}(a). \tag{3.2}$$

Let $\nu \in R_+$ be an arbitrary positive number, which can be obtained as the limit of the quotient of two subsequences of l. Let $\{\lambda_{n_k}\}_{k=1}^{\infty}$ and $\{\lambda_{m_k}\}_{k=1}^{\infty}$ be the two subsequences of l, for which

$$\nu = \lim_{k \to \infty} \frac{\lambda_{n_k}}{\lambda_{m_k}}$$

holds. Then theorem 1.1, (0.1) and (3.2) together imply

$$U_\nu(b) = \lim_{k \to \infty} U_{\frac{\lambda_{n_k}}{\lambda_{m_k}}} \left[U_{\lambda_{m_k}}(a)\right] = \lim_{k \to \infty} U_{\lambda_{n_k}}(a) = b$$

i.e. $\nu \in H(b)$. From this however by the definition of $Q(l)$ and by Theorem 2.1. there follows (3.1).

Theorem 3.2. *For any sequence $l \overset{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty} \in L$ there exists an operator $a \in M$, such that the operator sequence $\{U_{\lambda_n}(a)\}_{n=1}^{\infty}$ converges and*

$$H\left[\lim_{n \to \infty} U_{\lambda_n}(a)\right] = Q(l) \tag{3.4}$$

In proving this theorem it is necessary to distinguish three cases, according to the type of $Q(l)$. If $Q(l) = R_+$, then putting $a = 1$ (3.4) results trivially valid. If $Q(l) = \langle 1 \rangle$ or $Q(l)$ is a cyclic subgroup, the statement of Theorem 3.2 follows from Theorems 3.3 and 3.5, interesting by themselves.

Theorem 3.3. *For any operator $a \in M$ and for any sequence $l \overset{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty} \in L$ satisfying $Q(l) = \langle 1 \rangle$ there exists a continuous function $\varphi \in C$ such that the sequence $\{U_{\lambda_n}(\varphi)\}_{n=1}^{\infty}$ converges, and*

$$\lim_{n \to \infty} U_{\lambda_n}(\varphi) = a \tag{3.5}$$

PROOF. Let $a \in M$ be an arbitrary operator and $l \overset{\text{df}}{=} \{\lambda_n\}_{n=1}^{\infty} \in L$ an arbitrary sequence satisfying

$$Q(l) = \langle 1 \rangle \tag{3.6}$$

First we show that l has a subsequence $r \overset{\text{df}}{=} \{\varrho_n\}_{n=1}^{\infty}$ and there exists a sequence of the positive numbers $s \overset{\text{df}}{=} \{\sigma_n\}_{n=1}^{\infty}$, satisfying the following relations:

$$\varrho_1 < \varrho_2 < \ldots < \varrho_n < \varrho_{n+1} < \ldots, \tag{3.7}$$

$$\lim_{n\to\infty} \frac{\varrho_{n+1}}{\varrho_n} = +\infty, \tag{3.8}$$

$$\lim_{n\to\infty} \sigma_n = 1, \tag{3.9}$$

$$l = s \cdot r' \tag{3.10}$$

where r' is a subsequence of r.

Let

$$\varrho_1 \overset{\text{df}}{=} \lambda_1 \quad \varrho_{n+1} \overset{\text{df}}{=} \min\{\lambda_k \mid k = 1, 2, \ldots, \quad \lambda_k > 2\varrho_n\} \quad (n = 1, 2, \ldots) \tag{3.11}$$

Of course, the sequence r so defined is a subsequence of l. Let us define the sequence s as follows:

$$\sigma_n \overset{\text{df}}{=} \begin{cases} \dfrac{\lambda_n}{\varrho_1} & \text{if} \quad \lambda_n < \varrho_1, \quad \text{and} \\ \dfrac{\lambda_n}{\varrho_k} & \text{if} \quad \varrho_k \leqq \lambda_n < \varrho_{k+1} \end{cases} \qquad (n = 1, 2, \ldots) \tag{3.12}$$

Clearly, s is the quotient of l and of a subsequence of r, and so (3.10) holds. On the other hand (3.11) implies that 2 is a lower bound of the sequence $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$. From this (3.7) immediately follows, moreover, by (3.12) all but a finite number of elements of the sequence s fall into the interval [1, 2]. But on the basis of Definition 3.3 we infer from (3.6) that an accumulation point of the quotient of two subsequences fo l can only be 1 or 0. Since both s and $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$ are quotients of two subsequences of l, our above considerations directhy yield (3.8) and (3.9).

Let now

$$\frac{f}{g} = a \in M, \quad f, \quad g \in C \quad g \neq 0 \tag{3.13}$$

be an arbitrary representation of the operator a.

Let us define the sequence $m \overset{\text{df}}{=} \{\mu_n\}_{n=1}^{\infty}$ as follows:

$$\mu_n \overset{\text{df}}{=} \sqrt{\varrho_n \varrho_{n+1}} \quad (n = 1, 2, \ldots) \tag{3.14}$$

By (3.7) the sequence m so defined is a strictly monotonically increasing sequence of positive numbers, tending to infinity.

Let us now define the functions p, q and ψ on the half-line $0 \leqq t < \infty$ by induction with respect to the intervals determined by the members of the sequence m: let

$$p(t) \overset{\text{df}}{=} 1, \quad q(t) \overset{\text{df}}{=} 1 \quad \text{if} \quad 0 \leqq t \leqq \mu_1 \tag{3.15}$$

Since the function $\left\{\int_0^t q(\tau)\,d\tau\right\}=\{t\}$ $(0\leqq t\leqq\mu_1)$ does not vanish identically in any right neighborhood of 0, by the theorem of Foias, [1] there exists a continuous function ψ_1 defined on $0\leqq t\leqq\mu_1$ and satisfying the inequality

$$\int_0^{\mu_1}\left|\int_0^t \psi_1(t-\tau)\int_0^\tau q(\sigma)\,d\sigma\,d\tau-p(t)\right|dt<1 \tag{3.16}$$

Let

$$\psi(t)\overset{\mathrm{df}}{=}\psi_1(t), \quad \text{if} \quad 0\leqq t\leqq\mu_1. \tag{3.17}$$

Suppose that p, q and ψ have already been defined for some natural number n on the interval $0\leqq t\leqq\mu_n$. Let

$$p(t)\overset{\mathrm{df}}{=}c_{n+1}f\left(\frac{t}{\varrho_{n+1}}\right) \quad \text{and} \quad q(t)\overset{\mathrm{df}}{=}c_{n+1}g\left(\frac{t}{\varrho_{n+1}}\right) \quad \text{for} \quad \mu_n<t\leqq\mu_{n+1} \tag{3.18}$$

where

$$c_{n+1}\overset{\mathrm{df}}{=}\max\{|p(t)|,|q(t)|\,\big|\,0\leqq t\leqq\mu_n\} \tag{3.19}$$

By the theorem just mentioned of Foias, there again exists a continuous function ψ_{n+1} defined on the interval $0\leqq t\leqq\mu_{n+1}-\mu_n$ and satisfying the inequality

$$\int_0^{\mu_{n+1}-\mu_n}\left|\int_0^t \psi_{n+1}(t-\tau)\int_0^\tau q(\sigma)\,d\sigma\,d\tau-\right.$$
$$\left.-\left[p(t+\mu_n)-\int_0^{t+\mu_n}\psi(t+\mu_n-\tau)\int_0^\tau q(\sigma)\,d\sigma\,d\tau\right]\right|dt<\frac{1}{2^n} \tag{3.20}$$

Let

$$\psi(t)\overset{\mathrm{df}}{=}\psi_{n+1}(t-\mu_n) \quad \text{if} \quad \mu_n<t\leqq\mu_{n+1} \tag{3.21}$$

Also, let

$$\varphi\overset{\mathrm{df}}{=}\{1\}\psi \tag{3.22}$$

ψ is locally integrable, so $\varphi\in C$. Now we are going to show that the sequence $\{U_{\varrho_n}(\varphi)\}_{n=1}^\infty$ is convergent and has limit a.

Since q too was locally integrable, the sequences $\{U_{\varrho_n}(\{1\}q)/\varrho_n^2c_n\}_{n=1}^\infty$ and $\{U_{\varrho_n}(\varphi)\,U_{\varrho_n}(\{1\}q)/\varrho_n^2c_n\}_{n=1}^\infty$ $(c_1\overset{\mathrm{df}}{=}1)$ are, of course, sequences of continuous functions. We show that these sequences satisfy the following relations:

$$\frac{U_{\varrho_n}(\{1\}q)}{\varrho_n^2c_n}\rightrightarrows\{1\}g \quad (n\to\infty) \tag{3.23}$$

$$\frac{U_{\varrho_n}(\varphi)\,U_{\varrho_n}(\{1\}q)}{\varrho_n^2c_n}\rightrightarrows\{1\}f \quad (n\to\infty) \tag{3.24}$$

Let $T>0$ and $\varepsilon>0$ be arbitrary fixed numbers. (3.8) and (3.14) imply

$$\lim_{n\to\infty}\frac{\mu_n}{\varrho_n}=+\infty \quad\text{and}\quad \lim_{n\to\infty}\frac{\mu_n}{\varrho_{n+1}}=0 \tag{3.25}$$

and consequently there exists $N\geqq 2$ so that for $n\geqq N$ the following inequalities hold:

$$\varrho_n T<\mu_n,$$

$$\frac{1}{\varrho_n}\int_0^{\mu_{n-1}}\left|g\left(\frac{\tau}{\varrho_n}\right)\right|d\tau=\int_0^{\frac{\mu_{n+1}}{\varrho_n}}|g(\sigma)|\,d\sigma<\frac{\varepsilon}{2},$$

and

$$\frac{\mu_{n-1}}{\varrho_n}<\frac{\varepsilon}{2}$$

From this however it follows by (3.18) and (3.19) that if $n\geqq N$ then for $0\leqq t\leqq T$ the inequality

$$\left|\frac{1}{\varrho_n^2 c_n}\varrho_n\int_0^{\varrho_n t}q(\tau)\,d\tau-\int_0^t g(\sigma)\,d\sigma\right|\leqq\frac{1}{\varrho_n}\int_0^{\varrho_n t}\left|\frac{1}{c_n}q(\tau)-g\left(\frac{\tau}{\varrho_n}\right)\right|d\tau\leqq$$

$$\leqq\frac{1}{\varrho_n c_n}\int_0^{\mu_{n-1}}|q(\tau)|\,d\tau+\frac{1}{\varrho_n}\int_0^{\mu_{n-1}}|g(\tau)|\,d\tau+\frac{1}{\varrho_n c_n}\int_{\mu_{n-1}}^{\mu_n}\left|q(\tau)-c_n g\left(\frac{\tau}{\varrho_n}\right)\right|d\tau<\varepsilon$$

holds. Since T and ε have been arbitrary, (3.23) holds.

Let $T>0$ and $\varepsilon>0$ again be arbitrary fixed numbers. If we write $t-\mu_n$ from (3.20) instead of t and employ (3.21), then

$$\int_{\mu_n}^{\mu_{n+1}}\left|\int_0^t\psi(t-\tau)\int_0^\tau q(\sigma)\,d\sigma\,d\tau-p(t)\right|dt<\frac{1}{2^n}$$

results for any n, and by the associativity of convolution and by (3.22), (3.16) and (3.17) we infer that the following inequality is valid for any natural number n:

$$\int_0^{\mu_n}\left|\int_0^t\varphi(t-\tau)\,q(\tau)\,d\tau-p(t)\right|dt<2. \tag{3.26}$$

On the other hand, by (3.15) and (3.19) each element of the sequence $\{c_n\}_{n=1}^{\infty}$ is $\geqq 1$, and so, in view of $\lim_{n\to\infty}\varrho_n=+\infty$, we get

$$\lim_{n\to\infty}c_n\varrho_n=+\infty.$$

Owing to this and to (3.25) there exists an integer $N \geqq 2$ so that for $n \geqq N$ the following inequalities hold:

$$\varrho_n T < \mu_n,$$

$$\frac{1}{\varrho_n}\int_0^{\mu_{n-1}} \left|f\left(\frac{\tau}{\varrho_n}\right)\right| d\tau = \int_0^{\frac{\mu_{n-1}}{\varrho_n}} |f(\sigma)|\, d\sigma < \frac{\varepsilon}{3},$$

$$\frac{\mu_{n-1}}{\varrho_n} < \frac{\varepsilon}{3},$$

and

$$\frac{2}{c_n \varrho_n} < \frac{\varepsilon}{3}$$

From this we infer by (3.26), (3.18) and (3.19) that the following inequality is satisfied for any $n \geqq N$

$$\left|\int_0^t \varrho_n \varphi\big(\varrho_n(t-\tau)\big)\frac{1}{\varrho_n^2 c_n}\varrho_n \int_0^{\varrho_n \tau} q(\sigma)\, d\sigma\, d\tau - \int_0^t f(\tau)\, d\tau\right| \leqq$$

$$\leqq \frac{1}{c_n \varrho_n}\int_0^{\mu_n}\left|\int_0^{\tau} \varphi(\tau-\sigma)\, q(\sigma)\, d\sigma - p(\tau)\right| d\tau + \frac{1}{\varrho_n}\int_0^{\mu_n}\left|\frac{1}{c_n} p(\tau) - f\left(\frac{\tau}{\varrho_n}\right)\right| d\tau \leqq$$

$$\leqq \frac{2}{c_n \varrho_n} + \frac{1}{c_n \varrho_n}\int_0^{\mu_{n-1}} |p(\tau)|\, d\tau + \frac{1}{\varrho_n}\int_0^{\mu_{n-1}}\left|f\left(\frac{\tau}{\varrho_n}\right)\right| d\tau +$$

$$+ \frac{1}{c_n \varrho_n}\int_{\mu_{n-1}}^{\mu_n}\left|p(\tau) - c_n f\left(\frac{\tau}{\varrho_n}\right)\right| d\tau < \varepsilon \quad \text{for} \quad 0 \leqq t \leqq T$$

Since T and ε have been arbitrary, (3.24) holds. (3.23) and (3.24) together say that the sequence $\{U_{\varrho_n}(\varphi)\}_{n=1}^{\infty}$ is convergent, and by (3.13) it converges to the operator a. Now (3.9) and (3.10) imply by Theorem 1.1 and by (0.1) that the sequence $\{U_{\lambda_n}(\varphi)\}_{n=1}^{\infty}$ too is convergent, its limit being the same operator a. This completes the proof of Theorem 3.3.

As an interesting consequence of this theorem we obtain the following.

Theorem 3.4. *There exists a subring $\hat{C}$ of C, and an equivalence relation η defined on $\hat{C}$ and compatible with addition and convolution, so that the factor ring of $\hat{C}$ with respect to η is isomorphic with M.*

Proof. Let $l \overset{\text{df}}{=} \{n!\}_{n=1}^{\infty}$. Clearly, $l \in L$ and $Q(l) = \langle 1 \rangle$. Let $\hat{C}$ denote the set of all those functions φ from C, for which the sequence $\{U_{n!}(\varphi)\}_{n=1}^{\infty}$ converges. $\hat{C}$ is, of course, nonvoid, and by the additivity and the multiplicativity of the transformation U_λ, as well as by the fact that the sum and the product of convergent operator sequences is again convergent, the limit being the sum and the product of the original limits respectively, we irfer that $\hat{C}$ is a subring of C.

Let us define the mapping $\Phi : \hat{C} \to M$ as follows:

$$\Phi(f) \overset{\text{df}}{=} \lim_{n\to\infty} U_{n!}(f) \quad f \in \hat{C}$$

By the properties just mentioned of the transformation U_λ and of convergent sequences, Φ is operation-preserving with respect to both operations. Now by Theorem 3.3 Φ maps $\hat{C}$ onto M. Thus the relation η defined by

$$f\eta g \overset{\mathrm{df}}{\Leftrightarrow} \Phi(f) = \Phi(g) \quad (f, g \in \hat{C})$$

is a congruence on $\hat{C}$, and the factor-ring of $\hat{C}$ with respect to η results isomorphic with M.

Theorem 3.5. *Let $\alpha \in R_+ \alpha > 1$ be arbitrary. Then for any sequence $l \overset{\mathrm{df}}{=} \{\lambda_n\}_{n=1}^\infty \in L$ satisfying $Q(l) = \langle\alpha\rangle$, and for any operator $a \in M$ having a representation $a = f/g$ $(f, g \in C\ g \neq 0)$ satisfying*

$$f(\alpha t) = \alpha f(t), \quad g(\alpha t) = \alpha g(t)$$

for all $0 \leqq t < \infty$, there exists an operator $b \in M$ such that the operator sequence $\{U_{\lambda_n}(b)\}_{n=1}^\infty$ is convergent, and

$$\lim_{n\to\infty} U_{\lambda_n}(b) = a.$$

Proof. Let $\alpha \in R_+, \alpha > 1$ be an arbitrary number. Let $l \overset{\mathrm{df}}{=} \{\lambda_n\}_{n=1}^\infty \in L$ be an arbitrary sequence satisfying

$$Q(l) = \langle\alpha\rangle \tag{3.27}$$

Moreover, let $a \in M$ be an operator having a representation $a = f/g$ $(f, g \in C\ g \neq 0)$ satisfying

$$f(\alpha t) = \alpha f(t) \quad \text{and} \quad g(\alpha t) = \alpha g(t) \tag{3.28}$$

for $0 \leqq t < \infty$.

We now define the sequences $r \overset{\mathrm{df}}{=} \{\varrho_n\}_{n=1}^\infty$, $s \overset{\mathrm{df}}{=} \{\sigma_n\}_{n=1}^\infty$ and $m \overset{\mathrm{df}}{=} \{\mu_n\}_{n=1}^\infty$ as follows:

$$\varrho_1 \overset{\mathrm{df}}{=} \lambda_1, \quad \varrho_{n+1} \overset{\mathrm{df}}{=} \min\left\{\lambda_k \,|\, k = 1, 2, \ldots, \lambda_k > \frac{\alpha}{2}\varrho_n\right\} \quad (n = 1, 2, \ldots)$$

$$\sigma_n \overset{\mathrm{df}}{=} \begin{cases} \dfrac{\lambda_n}{\varrho_1} & \text{if} \quad \lambda_n < \varrho_1, \quad \text{and} \\ \dfrac{\lambda_n}{\varrho_k} & \text{if} \quad \varrho_k \leqq \lambda_n < \varrho_{k+1} \end{cases} \qquad (n = 1, 2, \ldots)$$

$$\mu_n \overset{\mathrm{df}}{=} \sqrt{\varrho_n \varrho_{n+1}} \quad (n = 1, 2, \ldots)$$

r is a subsequence of l, strictly monotonically increasing, and so m has the following properties:

$$0 < \mu_1 < \mu_2 < \ldots < \mu_n < \mu_{n+1} < \ldots \tag{3.29}$$

$$\lim_{n\to\infty} \mu_n = +\infty$$

Again, for the sequence s we have

$$s = \frac{l}{r'} \tag{3.30}$$

with r' a subsequence of r. On the other hand the definition of r implies that all but a finite number of elements of s fall into the interval $[1, \alpha/2]$, and since s is the quotient of two subsequences of l, (3.27) implies that

$$\lim_{n\to\infty} \sigma_n = 1. \tag{3.31}$$

From the definition of the sequence r we infer that the sequence $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$ has $\alpha/2$ as a lower bound, and since this sequence too is a quotient of two subsequences of l, (3.27) implies that α is an accumulation point of the sequence $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$, and that only powers of α with a positive integer exponent can be accumulation points of $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$.

Define now the functions p and q on the interval $0 \leqq t < \infty$ as follows:

$$p(t) \overset{\text{df}}{=} \varrho_1 f\left(\frac{t}{\varrho_1}\right),\ q(t) \overset{\text{df}}{=} \varrho_1 g\left(\frac{t}{\varrho_1}\right) \quad \text{if} \quad 0 \leqq t < \mu_1, \tag{3.32}$$

$$p(t) \overset{\text{df}}{=} \varrho_n f\left(\frac{t}{\varrho_n}\right),\ q(t) \overset{\text{df}}{=} \varrho_n g\left(\frac{t}{\varrho_n}\right) \quad \text{if} \quad \mu_{n-1} \leqq t < \mu_n \quad (n = 2, 3, \ldots).$$

This definition is correct by (3.29), and p, q are locally integrable functions. Let us now show that

$$\frac{1}{\varrho_n^3} U_{\varrho_n}(\{1\}p) \rightrightarrows \{1\}f \quad (n \to \infty) \tag{3.33}$$

$$\frac{1}{\varrho_n^3} U_{\varrho_n}(\{1\}q) \rightrightarrows \{1\}g \quad (n \to \infty) \tag{3.34}$$

Let $T>0$ and $\varepsilon>0$ be arbitrary fixed numbers. Let $\delta_1>0$ be so that

$$\left|\frac{1}{v} f(vt) - f(t)\right| < \frac{\varepsilon}{2T} \quad \text{and} \quad \left|\frac{1}{v} g(vt) - g(t)\right| < \frac{\varepsilon}{2T} \tag{3.35}$$

for any $\alpha^{-\delta_1} < v < \alpha^{\delta_1}$ and $0 \leqq t \leqq T$. Such a δ_1 exists by the unirofm continuity of the functions of two variables $\left\{\frac{1}{v} f(vt)\right\}$ and $\left\{\frac{1}{v} g(vt)\right\}$ on the closed rectangle $[1/\alpha, \alpha] \times [0, T]$. Now (3.35) directly implies the validity for $\alpha^{-\delta_1} < v < \alpha^{\delta_1}$ of the inequalities

$$\int_0^T \left|\frac{1}{v} f(vt) - f(t)\right| dt < \frac{\varepsilon}{2} \quad \text{and} \quad \int_0^T \left|\frac{1}{v} g(vt) - g(t)\right| dt < \frac{\varepsilon}{2} \tag{3.36}$$

On the other hand, from the continuity of f and of g there follows the existence of a number $\delta_2>0$, such that for $1/\alpha\leqq v\leqq\alpha$ the inequalities

$$(3.37)\qquad \int_0^{\delta_2}\left|\frac{1}{v}f(vt)-f(t)\right|dt<\frac{\varepsilon}{2}\quad\text{and}\quad\int_0^{\delta_2}\left|\frac{1}{v}g(vt)-g(t)\right|dt<\frac{\varepsilon}{2}$$

hold.

Let K be a positive number satisfying

$$(3.38)\qquad T<\alpha^K\quad\text{and}\quad\alpha^{-K}<\delta_2$$

Since only powers of α with a positive integer exponent can be points of accumulation of the sequence $\{\varrho_{n+1}/\varrho_n\}_{n=1}^{\infty}$, there exists an integer N so that for $n>N$

$$(3.39)\qquad \frac{\varrho_{n+1}}{\varrho_n}\in\left(\bigcup_{k=1}^{K}[\alpha^{k-\frac{\delta_1}{K}},\ \alpha^{k+\frac{\delta_1}{K}}]\right)\cup[\alpha^{K+1-\frac{\delta_1}{K}},\ \infty)$$

Let $n>N+K$ be an arbitrary fixed natural number. Without restricting generality, we can suppose $\delta_1<\frac{1}{2}$.

Then (3.39) implies the existence of a natural number $j\geqq N$, such that

$$(3.40)\qquad \frac{\varrho_{j+1}}{\varrho_n}<\alpha^{-K}$$

Let k be the natural number for which

$$(3.41)\qquad \varrho_j\leqq\varrho_{k-1}<\varrho_nT\leqq\varrho_k$$

Then by (3.38) and (3.39) there exists for any positive integer $j<i\leqq k$ a real number $\alpha^{-\delta_1}<v_i<\alpha^{\delta_1}$ and an integer n_i, so that

$$(3.42)\qquad \frac{\varrho_i}{\varrho_n}=\alpha^{n_i}v_i$$

for $j<i\leqq k$. Let moreover n_i and $1\leqq v_i<\alpha$ be an integer and a real number respectively, so that

$$(3.43)\qquad \frac{\varrho_i}{\varrho_n}=\alpha^{n_i}v_i\quad i=1,2,\ldots,j$$

Now (3.38), (3.40) and (3.41) imply that

$$\frac{\mu_j}{\varrho_n}=\frac{\sqrt{\varrho_j\varrho_{j+1}}}{\varrho_n}<\frac{\varrho_{k+1}}{\varrho_n}<\alpha^{-K}<\delta_2$$

and

$$T\leqq\frac{\varrho_k}{\varrho_n}<\frac{\sqrt{\varrho_k\varrho_{k+1}}}{\varrho_n}=\frac{\mu_k}{\varrho_n}$$

Then by (3.28), (3.32), (3.36), (3.37), (3.42) and (3.43) we have

$$\left|\frac{1}{\varrho_n^3}\varrho_n\int_0^{\varrho_n t}p(\tau)\,d\tau-\int_0^t f(\sigma)\,d\sigma\right|\leqq\frac{1}{\varrho_n}\int_0^{\varrho_n t}\left|\frac{1}{\varrho_n}p(\tau)-f\left(\frac{\tau}{\varrho_n}\right)\right|d\tau\leqq$$

$$\leqq\sum_{i=1}^{k-1}\int_{\frac{\mu_{i-1}}{\varrho_n}}^{\frac{\mu_i}{\varrho_n}}\left|v_i f\left(\frac{\sigma}{v_i}\right)-f(\sigma)\right|d\sigma+\int_{\frac{\mu_{k-1}}{\varrho_n}}^{T}\left|v_k f\left(\frac{\sigma}{v_k}\right)-f(\sigma)\right|d\sigma<\varepsilon.$$

In a similar manner, we obtain analogous inequalities for q and for g respectively. Since $n>N+K$ has been arbitrary, (3.33) and (3.34) follow, and this is equivalent to the statement that for $b\stackrel{\text{df}}{=}p/q$ the operator sequence $\{U_{\varrho_n}(b)\}_{n=1}^{\infty}$ converges, and

$$\lim_{n\to\infty}U_{\varrho_n}(b)=a$$

From this however, by (0.1) and Theorem 1.1, and in view of (3.31), we immediately get the statement of the theorem.

Definition 4.1. Let $l\stackrel{\text{df}}{=}\{\lambda_n\}_{n=1}^{\infty}\in L$ be an arbitrary sequence, and $a\in M$ an arbitrary operator. We say that a is l-integrable on the interval $(-\infty,\infty)$, if the operator sequence $\{U_{\lambda_n}(a)\}_{n=1}^{\infty}$ converges and has a number as its limit.

Definition 4.2. Let $a\in M$ be an arbitrary operator. We say that a is integrable on the interval $(-\infty,\infty)$, if it is l-integrable for any $l\in L$.

Theorem 4.1. *Let $l\stackrel{\text{df}}{=}\{\lambda_n\}_{n=1}^{\infty}\in L$ be an arbitrary sequence. l-integrability implies integrability, if and only if there exists a number $K>0$ such that for all n*

$$K\lambda_n\geqq\lambda_{k_n}>\lambda_n \tag{4.1}$$

holds for some k_n depending on n.

Proof. Let $l\stackrel{\text{df}}{=}\{\lambda_n\}_{n=1}^{\infty}\in L$ be a sequence satisfying (4.1). Let $r\stackrel{\text{df}}{=}\{\varrho_n\}_{n=1}^{\infty}\in L$ be an arbitrary sequence. On the basis of (4.1), for any natural number n there exists a natural number k_n, so that

$$K\lambda_{k_n}\geqq\varrho_n>\lambda_{k_n} \tag{4.2}$$

for any positive integer n, and $l'\stackrel{\text{df}}{=}\{\lambda_{k_n}\}_{n=1}^{\infty}$ is a subsequence of l. But by (4.2) the sequence r/l' has upper and lower bounds, and so Theorem 1.2 implies by (0.1) and on the basis of Definition 4.1 the sufficiency of the condition.

Let us now show that the condition is necessary. Let $l\stackrel{\text{df}}{=}\{\lambda_n\}_{n=1}^{\infty}\in L$ be a sequence which does not satisfy condition (4.1). Then for any natural number n there exists a natural number k_n $(k_n<k_{n+1},\ n=1,2,\ldots)$, so that

$$\lambda_k\notin(\lambda_{k_n},n^4\lambda_{k_n})\quad k=1,2,\ldots \tag{4.3}$$

Let p be the following function defined on $[0, \infty)$:

$$p(t) \stackrel{\text{df}}{=} \begin{cases} 0 & \text{if} \quad n\lambda_{k_n} < t < n^3\lambda_{k_n}, \quad \text{and} \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

Let $a \stackrel{\text{df}}{=} p/\{1\}$, and $r \stackrel{\text{df}}{=} \{n^2\lambda_{k_n}\}_{n=1}^{\infty} \in L$. We now show that the operator a is both l-integrable and r-integrable, but the two limits fail to coincide, and so a is not integrable.

Let $T>0$ and $\varepsilon>0$ be arbitrary numbers. Let n be a natural number, satisfying $n>T$ and $1/n<\varepsilon$. Let moreover K be a number, so that $k>K$ implies $\lambda_k>\lambda_{k_n}$. Then (4.3) and (4.4) together imply that for $k>K$ and $0 \leqq t \leqq T$ the following inequalities hold, where $m\,(\geqq n)$ denotes the positive integer for which $\lambda_{k_m}<\lambda_k<\lambda_{k_{m+1}}$:

$$\left|\frac{1}{\lambda_k^2}\lambda_k\int_0^{\lambda_k t} p(\tau)\,d\tau - t\right| \leqq \frac{1}{\lambda_k}\int_0^{m^3\lambda_{k_m}} |p(\tau)-1|\,d\tau + \frac{1}{\lambda_k}\int_{m^3\lambda_{k_m}}^{n\lambda_k} |1-1|\,d\tau \leqq \frac{1}{m} < \varepsilon,$$

$$\left|\frac{1}{(n^2\lambda_{k_n})^2}n^2\lambda_{k_n}\int_0^{n^2\lambda_{k_n}t} p(\tau)\,d\tau\right| \leqq \frac{1}{n^2\lambda_{k_n}}\int_0^{n^3\lambda_{k_n}} |p(\tau)|\,d\tau \leqq \frac{1}{n} < \varepsilon$$

Thus

$$\frac{1}{\lambda_n^2}U_{\lambda_n}(\{1\}p) \rightrightarrows \{t\} \quad (n \to \infty),$$

and

$$\frac{1}{(n^2\lambda_{k_n})^2}U_{(n^2\lambda_{k_n})}(\{1\}p) \rightrightarrows 0 \quad (n \to \infty)$$

i.e.

$$\lim_{n\to\infty} U_{\lambda_n}(a) = 1 \quad \text{and} \quad \lim_{n\to\infty} U_{n^2\lambda_{k_n}}(a) = 0.$$

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