Integrability conditions of a structure f_λ satisfying $f^3-\lambda^2 f=0$

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Summary: **Yano** [2] has obtained certain results on a structure defined by a tensor field $f\,(f\neq 0)$ of type (1,1) satisfying $f^3+f=0$. **Ishihara** and **Yano** [1] have obtained its integrability conditions. Now we propose to obtain integrability conditions of a structure f_λ satisfying $f^3-\lambda^2 f=0$, where λ is a complex number not equal to zero.

In section 2, we have studied the Nijenhuis tensor of f_λ—structure and deduced some of its properties. In sections 3 and 4, we have obtained the conditions of partial integrability and integrability of this structure in terms of its Nijenhuis tensor.

1. *Preliminaries:* Let M^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a (1,1) tensor field f $(f\neq 0)$ of class C^∞ satisfying:

$$f^3-\lambda^2 f=0, \tag{1.1}$$

where λ is a complex number not equal to zero.

Let us define the operators s and t by

$$s=\left(\frac{f}{\lambda}\right)^2,\quad t=I-\left(\frac{f}{\lambda}\right)^2, \tag{1.2}$$

I denoting the identity operator. Then we have

Theorem (1.1). *For a tensor field $f\neq 0$ satisfying* (1.1), *the operators s and t defined by* (1.2) *and applied to the tangent space at a point of the manifold are complementary projection operators.*

PROOF. By virtue of (1.1) and (1.2), we have

$$s+t=I.$$

$$s^2=\left(\frac{f}{\lambda}\right)^4=\frac{\lambda^2 f^2}{\lambda^4}=\left(\frac{f}{\lambda}\right)^2=s.$$

$$t^2=I+\left(\frac{f}{\lambda}\right)^4-2\left(\frac{f}{\lambda}\right)^2=1-\left(\frac{f}{\lambda}\right)^2=t.$$

$$st=ts=\left(\frac{f}{\lambda}\right)^2-\left(\frac{f}{\lambda}\right)^4=0.$$

Which proves the theorem.

Let S and T be the complementary distributions corresponding to the projection operators s and t respectively. If f is of constant rank r, then the dimensions of S and T are r and $(n-r)$ respectively. We call such a structure a *f_λ-structure of rank r* and the manifold M^n with this structure a *f_λ-manifold.*

Theorem (1.2). *For a tensor field $f\neq 0$ satisfying* (1.1) *and the operators s and t defined by* (1.2), *we have*

$$(1.3)\qquad \begin{aligned} fs &= sf = f, \quad ft = tf = 0. \\ f^2 s &= \lambda^2 s, \quad f^2 t = tf^2 = 0. \end{aligned}$$

i.e. f acts on S as a π-structure operator and on T as a null operator.

PROOF. By virtue of (1.1) and (1.2), we have

$$fs = sf = f\left(\frac{f}{\lambda}\right)^2 = \frac{f^3}{\lambda^2} = f.$$

$$f^2 s = f(fs) = f^2 = \lambda^2 s.$$

$$ft = tf = f - f\left(\frac{f}{\lambda}\right)^2 = 0.$$

$$f^2 t = tf^2 = f(ft) = 0.$$

Hence the result.

Corollary (1.1)*:* The f_λ-structure of maximal rank is a π-structure.

PROOF. If the rank of $f=n$, then $\dim T=0$ and $\dim S=n$. In this case, $t=0$ and $s=I$.

Hence f satisfies:

$$I-(f/\lambda)^2 = 0,$$

i.e.

$$f^2-\lambda^2 I = 0.$$

Hence the result.

2. *Nijenhuis tensor of f_λ-structure:*

For convenience henceforth, we will write f in place of f_λ-structure. Then the Nijenhuis tensor $N(X, Y)$ of f is

$$(2.1)\qquad N(X,Y) = [fX, fY]-f[fX, Y]-f[X, fY]+f^2[X, Y],$$

i.e.

$$(2.2)\qquad N(X,Y) = [fX, fY]-f[fX, Y]-f[X, fY]+\lambda^2 s[X, Y],$$

in consequence of (1.2).

Equation (2.1) can also be written in the form

$$(2.3)\qquad N(X,Y) = N(sX, sY)+N(sX, tY)+N(tX, sY)+N(tX, tY),$$

in consequence of (1.3).

Theorem (2.1). We have the following identities:

$$N(tX,tY) = s\cdot N(tX,tY) = \lambda^2 s[tX,tY], \tag{2.4}$$

$$t\cdot N(X,Y) = t\cdot[fX,fY], \tag{2.5}$$

$$t\cdot N(sX,sY) = t\cdot[fX,fY], \tag{2.6}$$

$$t\cdot N(fX,fY) = \lambda^4 t\cdot[sX,sY]. \tag{2.7}$$

Proof. The proofs of (2.4) to (2.7) follow by virtue of the equations (1.2), (1.3) and (2.2).

Theorem (2.2). *For any vector fields X and Y, the following conditions are equivalent:*

(i) $t\cdot N(X,Y)=0$,

(ii) $t\cdot N(sX,sY)=0$,

(iii) $t\cdot N(fX,fY)=0$.

Proof. In consequence of (1.2), (1.3) and (2.2), it can be easily proved that

$$N(sX,sY)=0 \quad \text{if and only if} \quad N(fX,fY)=0,$$

for all vector fields X and Y.

Thus, by virtue of (2.5) and (2.6), the conditions (i), (ii) and (iii) are equivalent to each other.

The Lie derivative $L_Y f$ of the tensor field f with respect to a vector field Y is, by definition, a tensor field of the same type as f given by

$$(L_Y f)X = f[X,Y]-[fX,Y]. \tag{2.8}$$

Then by virtue of (1.3), (2.1) and (2.8), we have

$$N(sX,tY) = f(L_{tY} f)sX = f\{s(L_{tY} f)sX\}, \tag{2.9}$$

i.e.

$$f\cdot N(sX,tY) = \lambda^2 s(L_{tY} f)sX, \tag{2.10}$$

in consequence of (1.2).

3. *Integrability conditions:*

Theorem (3.1). *For any two vector fields X and Y, the distribution T is integrable if and only if*

$$N(tX,tY)=0, \tag{3.1}$$

which is equivalent to

$$s\cdot N(tX,tY)=0. \tag{3.2}$$

PROOF. The distribution T is integrable if and only if

$$s\cdot[tX, tY] = 0,$$

for any two vector fields X and Y. Thus, by virtue of (2.4), the theorem follows.

Theorem (3.2). *The distribution S is integrable if and only if any one of the conditions of theorem* (2.2) *is satisfied.*

PROOF. The distribution S is integrable if and only if

$$t\cdot[sX, sY] = 0,$$

for any two vector fields X and Y. Therefore, by virtue of (2.7), the theorem follows.

Theorem (3.3). For any two vector fields X and Y, the distributions S and T are both integrable if and only if

$$N(X, Y) = s\cdot N(sX, sY) + N(sX, tY) + N(tX, sY). \tag{3.3}$$

PROOF. By virtue of $s+t=I$, equation (2.3) can also be written as

$$N(X, Y) = s\cdot N(sX, sY) + t\cdot N(sX, sY) + N(sX, tY) + + N(tX, sY) + N(tX, tY). \tag{3.4}$$

Thus by virtue of (3.1), (3.4) and theorem (3.2), we get the result.

Now suppose that the distribution S is integrable and take an arbitrary vector field X_1 tangent to an integral manifold of S. Let us define the operator f_* as follows:

$$f_* X_1 = fX_1.$$

Thus by virtue of theorem (1.2), the induced structure f_* is a π-structure on each integral manifold of S.

Let us denote by N_* (X_1, Y_1) the vector valued 2-form corresponding to the Nijenhuis tensor of the π-structure induced from f_λ-structure on each integral manifold of S and for any two vector fields X_1 and Y_1 tangent to an integral manifold of S. Then we have

$$N_*(X_1, Y_1) = [f_* X_1, f_* Y_1] - f_*[f_* X_1, Y_1] - - f_*[X_1, f_* Y_1] + \lambda^2[X_1, Y_1]; \tag{3.5}$$

from which in view of (2.1), it follows that

$$N(sX, sY) = N_*(sX, sY) \tag{3.6}$$

for any two vector fields X and Y in the manifold.

Definition (3.1): We say that the f_λ-structure is *partially integrable* if the distribution S is integrable and the π-structure f_* induced from f on each integral manifold of S is also integrable.

Theorem (3.4). For any two vector fields X and Y, a necessary and sufficient condition for the f_λ-structure to be partially integrable is that

$$N(sX, sY) = 0, \tag{3.7}$$

which is equivalent to

$$N(fX, fY) = 0. \tag{3.8}$$

PROOF. For any two vector fields X and Y, $N(sX, sY)=0$ if and only if $N(fX,fY)=0$. Therefore, by virtue of (3.6) and theorem (3.2), we obtain the result.

Theorem (3.5). For any two vector fields X and Y, a necessary and sufficient condition for the distribution T to be integrable and the f_λ-structure to be partially integrable is that

$$N(X, Y) = N(sX, tY)+N(tX, sY). \tag{3.9}$$

PROOF. The proof of the theorem follows by virtue of the equations (2.3), (3.1) and (3.7).

4. *The condition* $N(sX, tY)=0$.

Theorem (4.1). *The tensor field* $s(L_{tY}f)s$ *vanishes identically for any vector field* Y *if and only if*

$$N(sX, tY) = 0, \tag{4.1}$$

for any two vector fields X *and* Y.

PROOF. The proof of the theorem follows by virtue of the equations (2.9) and (2.10).

When both distributions S and T are integrable, we can choose a local coordinate system such that all S are represented by putting $(n-r)$ local coordinates constant and all T by putting the other r coordinates constant. Such a coordinate system will be called an 'adapted coordinate system'.

It can be supposed that in an adapted coordinate system, the projection operators s and t have the components of the form:

$$s = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \tag{4.2}$$

respectively, where I_r is a unit matrix of order r and I_{n-r} is of order $(n-r)$.

Since f satisfies: $fs=sf=f$ and $ft=tf=0$, the tensor f has the components of the form

$$f = \begin{pmatrix} f_r & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

in an adapted coordinate system, where f_r is a square matrix of order $r\times r$.

Thus, the Lie derivative $L_{tY} f$ has components of the form

$$L_{tY} f = \begin{pmatrix} L' & 0 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

for any vector field tY in T.

Theorem (4.2). *For both distributions S and T being integrable, the local components of the f_λ-structure are functions independent of the coordinates which are constant along the integral manifold of S in an adapted coordinate system if and only if, for any two vectvr fields X and Y*

$$N(sX, tY) = 0. \tag{4.5}$$

PROOF. Let us assume that $N(sX, tY)=0$ for any two vector fields X and Y. Then from theorem (4.1), the tensor field $s(L_{tY} f)s$ vanishes identically for any vector field Y. Hence we have

$$L_{tY} f = 0, \quad \text{and therefore} \quad L' = 0.$$

Which implies that the components of f are independent of the coordinates which are constant along the integral manifold of the distribution S in an adapted coordinate system.

Conversely, if the components of f are independent of these coordinates, then $L'=0$. Therefore the tensor field $s(L_{tY} f)s$ vanishes identically for any vector field Y. Hence $N(sX, tY)=0$ for any two vector fields X and Y.

Theorem (4.3). *Suppose that the distributions S and T are both integrable and that an adapted coordinate system has been chosen. Then the components of f are independent of the coordinates which are constant along the integral manifold of S if and only if*

$$N(X, Y) = s \cdot N(sX, sY), \tag{4.6}$$

for any two vector fields X and Y.

PROOF. The proof of the theorem follows by virtue of (3.3) and theorem (4.2).

Definition (4.1) We say that the f_λ-structure is 'integrable' if

(i) the structure f_λ is partially integrable: i.e.,

$$N(sX, sY) = 0;$$

(ii) the distribution T is integrable: i.e.,

$$N(tX, tY) = 0; \quad \text{and}$$

(iii) the components of the f_λ-structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system.

Theorem (4.4). *A necessary and sufficient condition for the structure f_λ to be integrable is that*

$$N(X, Y) = 0, \tag{4.7}$$

for any two vector fields X and Y.

PROOF. The proof of the theorem follows from the theorems (3.1), (3.4) and (4.3).

References

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