On some functional equations in Banach algebras

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Let R denote the field of real numbers, $\mathbf{B}$ a complex Banach algebra, f, g functions mapping R into $\mathbf{B}$. The purpose of this note is to study the solutions of the functional equations

$$f(\xi+\eta)+f(\xi-\eta) = 2f(\xi)\,g(\eta) \tag{1}$$

(cf. e.g. [1], [7]) and

$$f(\xi+\eta)\cdot f(\xi-\eta) = f(\xi)^2-f(\eta)^2 \tag{2}$$

(see. e.g. [1], [5], [6]) under certain measurability and continuity conditions. We stipulate that φ_e and φ_0 denote the even and odd parts, respectively, of the function φ throughout.

Lemma 1. *If f and g satisfy* (1) *and are strongly measurable, then f is strongly continuous. Let $A_r(H)$ denote the right annihilator of $H=\{f(\xi);\,\xi\in R\}$, and suppose $A_r(H)=\{0\}$. Then g is continuous, even and satisfies*

$$g(\xi+\eta)+g(\xi-\eta) = 2g(\xi)\,g(\eta). \tag{3}$$

PROOF. Since (1) can be written as $f(\eta)=2f\left(\frac{\xi+\eta}{2}\right)g\left(\frac{-\xi+\eta}{2}\right)-f(\xi)$, the first statement follows at once from [2], Theorem 2. From (1) we get $f(\xi)g(\eta)= =f(\xi)\cdot g(-\eta)$, which implies $f(\xi)g_0(\eta)=0$.

Then, by assumption, $g_0(\eta)=0$ for $\eta\in R$, thus g is even. From (1) we obtain for $\alpha, \xi, \eta\in R$

$$\begin{aligned} f(\alpha)\{g(\xi+\eta)+g(\xi-\eta)\} &= \frac{1}{2}\{f(\alpha+\xi+\eta)+ \\ &+f(\alpha-\xi-\eta)+f(\alpha+\xi-\eta)+f(\alpha-\xi+\eta)\} = \\ &= \{f(\alpha+\xi)+f(\alpha-\xi)\}\,g(\eta) = 2f(\alpha)\,g(\xi)\,g(\eta). \end{aligned} \tag{4}$$

This, by assumption, implies (3) and, by the measurability of g, that g is continuous.

Moreover, we have

Theorem 1. *Suppose f, g are continuous and satisfy* (1), *and g satisfies* (3) *with $g(0)=j$. Then j is idempotent and there exist uniquely determined elements $a, b, c, d\in B$ such that $a=aj$, $b=bj=jb$, $c=cj$, $jc=0$, $d=dj$ and*

$$f(\xi) = a\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)+ \tag{5}$$

$$+d\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right),$$

$$g(\xi) = \left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)+ \tag{6}$$

$$+c\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right).$$

The converse is also true.

Proof. Since g satisfies (3), the statements concerning g, j, b and c follow from [3]. Moreover, $f(\xi)g_0(\eta)=0$ implies

$$f_e(\xi+\eta)+f_e(\xi-\eta)+f_0(\xi+\eta)+f_0(\xi-\eta)= \tag{7}$$
$$=2\{f_e(\xi)+f_0(\xi)\}g_e(\eta)$$

and thus

$$f_e(\xi+\eta)+f_e(\xi-\eta) = 2f_e(\xi)g_e(\eta) = 2f_e(\eta)g_e(\xi). \tag{8}$$

From (8) we get $f_e(\xi)=f_e(\xi)j=f_e(0)g_e(\xi)$, which gives with $f_e(0)=a$ that

$$a = aj \quad\text{and}\quad f_e(\xi) = a\cdot\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right). \tag{9}$$

From (7) and (8) we obtain

$$f_0(\xi+\eta)+f_0(\xi-\eta) = 2f_0(\xi)g_e(\eta). \tag{10}$$

Since $jg_e(\eta)=g_e(\eta)j=g_e(\eta)$ for $\eta\in R$, thus $g_e(\eta)$ belongs to the closed subalgebra $\mathbf{B}_j=\{x\in\mathbf{B}:xj=x=jx\}$ with unit j (cf. e.g. [3]), and for some sufficiently small $\varepsilon>0$ there exists the inverse r in $\mathbf{B}_j$ of $\int_0^\varepsilon g_e(\eta)\,d\eta$. Integrating (10) we get, by (10) with $\eta=0$,

$$f_0(\xi) = f_0(\xi)\cdot j = \frac{1}{2}\cdot\int_{\xi-\varepsilon}^{\xi+\varepsilon} f_0(\tau)\,d\tau\cdot r \tag{11}$$

thus f_0 has continuous derivatives of any order on R. Differentiating (10) twice we get $f_0''(\xi+\eta)+f_0''(\xi-\eta)=2f_0(\xi)g_e''(\eta)$, which gives

$$f_0''(\xi) = f_0(\xi)\cdot b \tag{12}$$

with $f_0(0)=0$ and, in view of (11), $d=f_0'(0)=d\cdot j$. The solution of (12) under these conditions can be obtained as in [3], and is $f_0(\xi)=d\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right)$. Thus we get (5), and since the converse is straightforward, the proof is complete.

Corollary. If f, g are strongly measurable, satisfy (1), and $A_r(H)=\{0\}$, then the conclusions of Theorem 1 hold with $c=0$.

The general measurable solutions of (1) can be somewhat pathological, and in the case of a commutative Banach algebra can be described as follows.

Theorem 2. *Suppose* $\mathbf{B}$ *is a commutative Banach algebra,* f, g *are strongly measurable and satisfy* (1), *further* N *denotes the annihilator of* $H=\{f(\xi);\ \xi\in R\}$. *Then there exist uniquely determined elements* $j, a, d\in\mathbf{B}$ *with* $j^2-j\in N, aj=a, dj=d$ *and* $b\in\mathbf{B}$ *determined modulo* N *with* $bj-b\in N$ *such that*

$$f(\xi)=a\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)+d\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right) \tag{13}$$

$$g(\xi)=\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)+r(\xi;\ b) \tag{14}$$

with $r(0;b)=0$, *where* r *is a stongly measurable map of* R *into* N. *Conversely, if* $j, b, a, d\in\mathbf{B}$ *with* $aj=a, dj=d, N$ *is the annihilator of* $\{a\}\cup\{d\}$, r *is a map of* R *into* N, *and* f, g *are given by* (13) *and* (14), *then* f, g *satisfy* (1).

Proof. By assumption, N is a closed ideal in $\mathbf{B}$. From (4) we see that $g(\xi+\eta)+g(\xi-\eta)-2g(\xi)g(\eta)\in N$ for $\xi, \eta\in R$. Let $\mathbf{B}/N$ denote the quotient algebra with norm $\|X\|=\inf\{\|x\|;\ x\in X\}$ and Q the quotient map of $\mathbf{B}$ onto $\mathbf{B}/N$. Then $\mathbf{B}/N$ is also a commutative Banach algebra and the map $G=Q\circ g$ of R into $\mathbf{B}/N$ is strongly measurable and satisfies (3). According to [3] there exist uniquely determined elements $J, B\in\mathbf{B}/N$ such that $G(\xi)=J+B\frac{\xi^2}{2!}+B^2\frac{\xi^4}{4!}+\dots$ with $J^2=J$, $BJ=B$. If we put $g(0)=j\in J$, and choose an element $b\in B$, we have $j^2-j\in N$, $bj-b\in N$ and that $r(\xi;b)=g(\xi)-\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)\in N$, where r is strongly measurable with $r(0;b)=0$. Introducing the notation $g_d(\xi)=j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots$, we get from (1) as in the proof of Theorem 1 $f_e(\xi)=f(0)g_d(\xi)=f_e(\xi)j$, which gives with $f(0)=a$ $a=aj$ and $f_e(\xi)=a\left(j+b\frac{\xi^2}{2!}+b^2\frac{\xi^4}{4!}+\dots\right)$. Similarly, we obtain

$$f_0(\xi+\eta)+f_0(\xi-\eta)=2f_0(\xi)\,g_d(\eta) \tag{15}$$

and hence

$$\frac{1}{2}\cdot\int_{\xi-\varepsilon}^{\xi+\varepsilon} f_0(\tau)\,d\tau=f_0(\xi)\cdot\int_0^{\varepsilon} g_d(\eta)\,d\eta\quad(\varepsilon>0). \tag{16}$$

With the notation $x(\varepsilon)=\frac{1}{\varepsilon}\int_0^{\varepsilon} g_d(\eta)\,d\eta$ we have $\lim_{\varepsilon\to0} x(\varepsilon)=j$, thus with $X(\varepsilon)=Qx(\varepsilon)$ we get $\lim_{\varepsilon\to0} X(\varepsilon)=J$. If $\mathbf{B}_J=\{X\in\mathbf{B}/N: XJ=X\}$, then $\mathbf{B}_J$ is a closed commutative subalgebra of $\mathbf{B}/N$ with unit J. Since $g_d(\eta)\cdot j=j+m+(b+n)\frac{\eta^2}{2!}+b(b+n)\frac{\eta^4}{4!}+$

$+b^2(b+n)\frac{\eta^6}{6!}+\ldots=g_d(\eta)+p(\eta)$, where $m=j^2-j\in N$, $n=bj-b\in N$, $p(\eta)\in N$ for $\eta\in R$, we have $x(\varepsilon)\cdot j=\frac{1}{\varepsilon}\int\limits_0^\varepsilon g_d(\eta)\cdot j\,d\eta=x(\varepsilon)+\frac{1}{\varepsilon}\int\limits_0^\varepsilon p(\eta)\,d\eta\in X(\varepsilon)$, for N is closed. Thus $X(\varepsilon)\in \mathbf{B}_J$ for every $\varepsilon>0$, and for some $\varepsilon>0$ there exists a $Y(\varepsilon)\in\mathbf{B}_J$ with $X(\varepsilon)Y(\varepsilon)=J$, consequently there exists a $y(\varepsilon)\in\mathbf{B}$ with $x(\varepsilon)y(\varepsilon)-j=q\in N$. It follows that $\int\limits_0^\varepsilon g_d(\eta)\,d\eta\cdot\frac{1}{\varepsilon}y(\varepsilon)=j+q$, and (15) and the definition of N give $f_0(\xi)\cdot(j+q)=f_0(\xi)$. Thus (16) implies $f_0(\xi)=\frac{1}{2\varepsilon}y(\varepsilon)\int\limits_{\xi-\varepsilon}^{\xi+\varepsilon}f_0(\tau)\,d\tau$, consequently f_0 has derivatives of every order on R. A similar argument as in the proof of Theorem 1 yields that with $d=f_0'(0)=f_0'(0)j=dj$ we have

$$f_0(\xi)=d\left(j\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\ldots\right),$$

while the converse part of the theorem can be proved by straightforward calculation.

In what follows we deal with the equation (2). We always assume that $\mathbf{B}$ is an algebra with unit e and, to avoid pathological phenomena (cf. [5]) that $f(\alpha)$ is regular for some $\alpha\in R$.

Lemma 2. *Suppose* $\mathbf{B}$ *is an algebra with unit* e, $f:R\to\mathbf{B}$ *satisfies* (2) *and for some* $\alpha\in R$ $f(\alpha)^{-1}$ *exists. Then* f *is odd,* $g(\xi)=\frac{1}{2}f(\alpha)^{-1}\{f(\xi+\alpha)-f(\xi-\alpha)\}$ *satisfies* (3) *with* $g(0)=e$, *and* f, g *satisfy* (1).

PROOF. If we put (η,ξ) instead of (ξ,η) in (2), we get $f(\alpha)\{f(\xi)+f(-\xi)\}=0$ for $\xi\in R$. Thus f is odd, and this implies that $f(\xi), f(\eta)$ commute for $\xi,\eta\in R$ (cf. [5]). Consequently, $f(\alpha)^{-1}$ also commutes with $f(\xi)$ for every $\xi\in R$, and the calculation in [1], p. 137 applies and gives that g satisfies (3) with $g(0)=e$. Moreover, a calculation similar to that in [1], p. 138 yields that f, g satisfy (1), and the lemma is proved.

The following theorem generalizes some results of S. KUREPA (cf. [5]).

Theorem 3. *Suppose* $\mathbf{B}$ *is a Banach algebra with unit* e, $f:R\to\mathbf{B}$ *satisfies* (2) *and is strongly measurable on a set* $P\subset R$ *of positive Lebesgue measure, further* $f(\alpha)$ *is regular for some* $\alpha\in R$. *Then there exist uniquely determined elements* $b, d\in\mathbf{B}$ *with* $bd=db$ *such that* $f(\xi)=d\left\{e\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\ldots\right\}$. *Conversely, if* f *is of the above form with* $bd=db$, *then* f *satisfies* (2).

PROOF. We first show that under these conditions 0 is in the closure of the set $H=\{\xi\in R: f(\xi)\text{ regular}\}$. Indeed, according to Lemma 2 we have for every $\xi\in R$ $f(2\xi)=2f(\xi)g(\xi)$, where $f(\xi)$ and $g(\xi)$ commute. If we assume that $f(\xi)$ is singular for $|\xi|<\varepsilon$ with $\varepsilon>0$, then it follows that $f(\xi)$ is singular also for $|\xi|<2\varepsilon$ and thus for every $\xi\in R$, which contradicts the assumptions of the theorem.

Moreover, according to [4] the map $\varphi:R^2\to[0,\infty)$, $\varphi(\xi_1,\xi_2)=m\{(P+\xi_1)\cap\cap(P+\xi_2)\}$ (m denotes Lebesgue measure) is continuous with $\varphi(0,0)=m(P)>0$.

By the above reasoning, for some $\alpha \in H$ we have $m(P_\alpha)=m\{(P+\alpha)\cap(P-\alpha)\}>0$. $\xi \in P_\alpha$ implies $\xi+\alpha,\ \xi-\alpha \in P$, consequently $g(\xi)=\frac{1}{2}f(\alpha)^{-1}\{f(\xi+\alpha)-f(\xi-\alpha)\}$ is measurable on P_α and [3], Corollary to Prop. 2. ensures the continuity of g. Moreover, there exists a compact $K\subset P$ such that $m(K)>0$, and the restriction of f to K is uniformly continuous (cf. e.g. [2]), while the restriction of g to K is also uniformly continuous.

Suppose $\varepsilon>0$ and find a $\delta>0$ such that $|\eta|<\delta$ implies $L_\eta=K\cap(K+\eta)\cap\cap(K-\eta)\neq\emptyset$ and that $\|f(\xi)-f(\xi+\eta)\|<\varepsilon$ and $\|g(\xi)-g(\xi+\eta)\|<\varepsilon$ whenever ξ, $\xi+\eta\in K$. Since f, g satisfy (1), for $\xi, \eta\in R$ we have $f(2\eta)=2f(\xi+\eta)g(\xi-\eta)-f(2\xi)$ and, applying again (1)

$$f(2\eta) = 2f(\xi+\eta)g(\xi-\eta)-2f(\xi)g(\xi) \quad (\xi,\eta\in R). \tag{17}$$

If $|\eta|<\delta$, then there exists a $\xi\in L_\eta$, for which we have $\xi, \xi-\eta, \xi+\eta\in K$ and, putting $S=\max\{\|f(\xi)\|, \|g(\xi)\|: \xi\in K\}$ we obtain by (17)

$$\frac{1}{2}\|f(2\eta)\| \leqq \|f(\xi+\eta)\|\cdot\|g(\xi-\eta)-g(\xi)\|+$$
$$+\|f(\xi+\eta)-f(\xi)\|\cdot\|g(\xi)\| \leqq 2S\varepsilon,$$

thus f is continuous at 0. Since f is odd, we get

$$f(\xi+\eta)-f(\xi)=f(\xi+\eta)+f(-\xi)=$$
$$=2f(\eta/2)g(\xi+\eta/2) \quad (\xi,\eta\in R)$$

and the continuity of g implies the continuity of f on R.

Now we apply Theorem 1 and establish by (5) that $f(\xi)=d\left(e\xi+b\frac{\xi^3}{3!}+b^2\frac{\xi^5}{5!}+\dots\right)$ with $d=f'(0)$. Moreover, since $f(\xi)$ and $g(\eta)$ commute for $\xi,\eta\in R$ we also obtain $f'(\xi)\cdot g''(\eta)=g''(\eta)\cdot f'(\xi)$ and, putting $\xi=\eta=0$, $db=bd$. The converse can be obtained by direct calculation, thus the theorem is proved.

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