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Mean-square derivative of a linear non-anticipative transformation of a continuous martingale

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Abstract. We give necessary and sufficient conditions so that linear non-anticipating process over the continuous martingale is differentiable with respect to structural function of underlying martingale. This result is applied to the linear theory of non-anticipating processes and a series of consequences is derived.

1. Let $Z = \{Z(t), t \geq 0\}$ be a real mean-square continuous random process, $\mathbb{E}Z(t) = 0, Z(0_+) = 0$. Denote by $\mathcal{H}(Z;t)$ the mean-square linear closure of $\{Z(s), s \leq t\}$, $\mathcal{H}(Z) = \overline{\bigvee_t \mathcal{H}(Z;t)}$, and by P_t^Z an orthogonal projection operator onto $\mathcal{H}(Z,t)$. In the sequel we shall suppose that Z is *wide-sence martingale*, i.e. $P_s^Z Z(t) = Z(s)$, $0 \leq s < t$ with the *structural function* $F(t) = \mathbb{E}Z^2(t) = \|Z(t)\|^2$, $t \geq 0$. It is well-known that any $\xi \in \mathcal{H}(Z;t)$ has the representation

$$\xi = \int_0^t g(u) Z(du), \quad \|\xi\|^2 = \int_0^t g^2(u) F(du) \tag{1}$$

for some $g \in L_2([0,t]; F(du))$.

A process $X = \{X(t), t \geq 0\}$ is the *linear non-anticipative transformation* of Z if $X(t) \in \mathcal{H}(Z;t)$ for any $t \geq 0$. It follows immediately from

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(1) that X has the representation

$$X(t) = \int_0^t g(t,u)Z(du), \quad \|X(t)\|^2 = \int_0^t g^2(t,u)F(du). \tag{2}$$

We also use notation $X = [g, Z]$.

Definition 1. Mean-square derivative of X is the process $\dot{X} = \{\dot{X}(t), t > 0\}$ defined by

$$X(t) = \int_0^t \dot{X}(s)F(ds), \quad \int_0^t \mathbb{E}(\dot{X}(s))^2 F(ds) < \infty. \tag{3}$$

We assume that $\dot{X}(t_0) = 0$ if $t = t_0$ is not the increasing point for F.

It is easy to see that the existence of $\dot{X}(t) \neq 0$ implies the existence of

$$X'(t) = \mathrm{l.\,i.\,m.}_{h\to 0} \frac{X(t+h) - X(t)}{F(t+h) - F(t)}, \quad \text{and} \quad X'(t) = \dot{X}(t), \tag{4}$$

but it is possible that X' exists and $\dot{X}$ does not exists.

The following example motivated us to regard the derivative of X as the process $\dot{X}$.

Example 1. Consider the case $Z = W$, where $W = \{W(t), t \geq 0\}$ is a wide sense Wiener process ($F(dt) = dt$). It follows from (4) that X' is non-anticipative transformation of W, say $X' = [f, W]$. For $h > 0$ we have

$$\begin{aligned}
&\Big\| \frac{X(t+h) - X(t)}{h} - X'(t) \Big\|^2 \\
&= \Big\| \int_0^t \Big(\frac{g(t+h,u) - g(t,u)}{h} - f(t,u) \Big) W(du) \Big\|^2 \\
&\quad + \Big\| \int_t^{t+h} \Big(\frac{g(t+h,u)}{h} \Big) W(du) \Big\|^2 \\
&= \int_0^t \Big(\frac{g(t+h,u) - g(t,u)}{h} - f(t,u) \Big)^2 du + \int_t^{t+h} \Big(\frac{g(t+h,u)}{h} \Big)^2 du,
\end{aligned}$$

so $X'(t)$ exists if and only if the two last summands tend to 0 when $h \to 0$.

Put

$$X(t) = \int_0^t (K(t) - K(u))W(du), \quad 0 \leq t \leq 1,$$

where $K(x), 0 \le x \le 1$, is Cantor distribution function. As $K'(x) \overset{\text{a.e.}}{=} 0$, we have

$$\begin{aligned}
&\Big\|\frac{X(t+h)-X(t)}{h}\Big\|^2 \\
&= \frac{1}{h^2}\Big\{\int_0^t (K(t+h)-K(t))^2 du + \int_t^{t+h} (K(t+h)-K(u))^2 du\Big\} \\
&\le \frac{t}{h^2}(K(t+h)-K(t))^2 + \frac{1}{h^2}\int_t^{t+h} (K(t+h)-K(t))^2 du \\
&= (t+h)\Big\{\frac{1}{h}(K(t+h)-K(t))\Big\}^2 \to 0, \quad h \to 0,
\end{aligned}$$

so $X'(t) = 0$ and the process X is not reproducible by the process X'. $\square$

Let us return to the general case (3). it follows from (4) that $\dot{X}$ is non-anticipative transformation, $\dot{X} = [f, Z]$.

Proposition 1. *The process $X = [g, Z]$ has the derivative $\dot{X} = [f, Z]$ if and only if*

$$g(t,u) = \int_u^t f(x,u)F(dx). \tag{5}$$

Proof. Let t be a point of increase of F. We have

$$\begin{aligned}
&\Big\|X(t) - \int_0^t \dot{X}(s)F(ds)\Big\|^2 \\
&= \Big\|\int_0^t g(t,u)Z(du) - \int_0^t \Big(\int_0^s f(s,u)Z(du)\Big) F(ds)\Big\|^2 \\
&= \Big\|\int_0^t g(t,u)Z(du) - \int_0^t \Big(\int_u^t f(s,u)F(ds)\Big) Z(du)\Big\|^2 \\
&= \int_0^t \big(g(t,u) - \int_u^t f(s,u)F(ds)\big)^2 F(du),
\end{aligned}$$

and the conclusion follows immediately. $\square$

As an example connected to the previous discussion, consider $X = [g, Z]$ where the structural function of Z is $K(t)$ and $g(t,u) = K(t) - K(u) = \int_u^t 1\, K(dx)$. Then $\dot{X} = \int_0^t 1\, Z(du) = Z(t)$, i.e. $\dot{X} = [1, Z]$.

The derivative process $\dot{X}$ does not depend on the representation $X = [g, Z]$ in the following sense:

Proposition 2. *Let $\mathcal{H}(Z_1;t) = \mathcal{H}(Z_2;t)$ for each t, and let X have corresponding representations $[g_1, Z_1]$ and $[g_2, Z_2]$. Then the processes $\dot{X}_1$ and $\dot{X}_2$ coincide.*

Proof. As for each t we have $\mathcal{H}(Z_1;t) = \mathcal{H}(Z_2;t)$ it follows that the measures F_1 and F_2 are equivalent. Let

$$\frac{F_1(du)}{F_2(du)} = \phi(u) > 0 \quad \text{a.e.} \quad F_2(du)$$

be the Radon-Nikodym derivative. Then

$$X(t) = \int_0^t g_1(t,u) Z_1(du) = \int_0^t g_1(t,u)\sqrt{\phi(u)} Z_2(du)$$

so $g_2(t,u) = g_1(t,u)\sqrt{\phi(u)}$ and $f_2(t,u) = f_1(t,u)\sqrt{\phi(u)}$, and

$$\begin{aligned}\dot{X}_1(t) &= \int_0^t f_1(t,u) Z_1(du) = \int_0^t \frac{f_2(t,u)}{\sqrt{\phi(u)}}\sqrt{\phi(u)} Z_2(du)\\ &= \int_0^t f_2(t,u) Z_2(du) = \dot{X}_2(t). \qquad \square\end{aligned}$$

For example, if in (2) the measure F is equivalent to the Lebesgue measure ($F(dt) \backsim dt$) the Proposition 2 allows us to consider W instead of Z, where W is a wide-sense Wiener process. In that case X have representation

$$X(t) = \int_0^t g(t,u) W(du), \quad \|X(t)\|^2 = \int_0^t g^2(t,u)\,du. \tag{6}$$

For the sake of simplicity in the rest of the paper we shall deal with the representations in the form (6).

Now it is easy to prove the following

Proposition 3. *If a non-zero process $\{Y(t), t \geq 0\}$ is mean-square analytic then it is not a non-anticipative transformation of a Wiener process.*

Proof. Let

$$Y(t) = \int_0^t g_0(t,u) W(du).$$

Then

$$\dot{Y}^{(n)}(t) = \int_0^t g_n(t,u) W(du) \quad (n \geq 0)$$

where

$$g_n(t,u) = \int_u^t g_{n+1}(z,u)dx \quad (n \geq 0).$$

It is evident that $g_n(u,u) = 0$. Expanding $g_0(t,u)$ in Taylor series with respect to t in the neighborhood of $t = u$ we get

$$g_0(t,u) = g_0(u,u) + \frac{(t-u)}{1!}g_1(u,u) + \frac{(t-u)^2}{2!}g_2(u,u) + \cdots = 0$$

for each $(t,u), u \leq t$. $\square$

Example 2. Consider Loève-Karhunen representation of $\{W(t),\ 0 \leq t \leq 1\}$:

$$W(t) = \sum_{n=0}^{\infty} Z_n \int_0^t \phi_n(s)ds$$

where $\{Z_n, n \geq 0\}$ is orthonormal basis in $\mathcal{H}(W)$ and

$$Z_n = \int_0^1 \phi_n(s)W(ds).$$

Let $\dot{Y}^{(n)}(t_0) = Z_n$, and define mean-square analytic process $\{Y(t), 0 \leq t \leq 1\}$ as

$$Y(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} Z_n$$

It is evident that $Y(t) \in \mathcal{H}(W)$ for each $t \in [0,1]$, but $Y(t) \notin \mathcal{H}(W;t)$ for any $t \in [0,1)$. So $\{Y(t)\}$ is not a non-anticipative transformation of $\{W(t)\}$.

Remark 1. Now we shall consider, more generally, the nonanticipative non-linear transformation of Wiener process $\{W(t), t \geq 0\}$. (Not a wide sense Wiener process.) Let $\mathcal{H}(W;t)$ be the mean-square linear closure of all random variables (of finite variance) measurable with respect to σ-field generated by $\{W(u), u \leq t\}$,$\mathcal{H}(W) = \overline{\bigvee_t \mathcal{H}(W;t)}$. It was shown in [3] that there exist mutually orthogonal wide sense Wiener processes $\{H_p(t), t \geq 0\}$, $p = 1, 2, \ldots$, $(H_1(t) = W(t))$ such that $H_p(t) \in \mathcal{H}(W;t)$ for each t and p, and

$$\mathcal{H}(W;t) = \oplus \sum_{p=1}^{\infty} \mathcal{H}(H_p;t)$$

for each t. The process $\{\mathbb{X}(t), t \geq 0\}$ is a *non-linear non-anticipative transformation of a Wiener process* $\{W(t)\}$ if $\mathbb{X}(t) \in \mathcal{H}(W;t)$ for each t. The representation

$$\mathbb{X}(t) = \sum_{p=1}^{\infty} \int_0^t g_p(t,u) H_p(du), \quad \|\mathbb{X}(t)\|^2 = \sum_{p=1}^{\infty} \int_0^t g_p^2(t,u)du$$

follows immediately. So we have

Proposition 4. *Suppose that* $g_p(t,u) = \int_u^t f_p(x,u)dx$ *for each* $p = 1, 2, \dots$ *and*

$$\sum_{p=1}^{\infty} \int_0^t f_p(t,u) H_p(du),$$

converges locally uniformly in mean-square sense over $t > 0$. *Then* $\dot{\mathbb{X}}(t)$ *exists and*

$$\dot{\mathbb{X}}(t) = \sum_{p=1}^{\infty} \int_0^t f_p(t,u) H_p(du). \qquad \square$$

In the remaining part of the paper we shall consider only non-anticipative linear transformations, so the adjective 'linear' will be omitted.

2. According to Hida [3], the representation (6) is canonical if the projection $\overline{X}(t,s) = P_t^X X(t)$ of $X(t)$ onto $\mathcal{H}(X;s)$, $s < t$, is of the form

$$\overline{X}(t,s) = \int_0^s g(t,u) W(du). \tag{7}$$

The representation (6) is proper canonical if $\mathcal{H}(X,t) = \mathcal{H}(W,t)$ for each t. We have the following situation in terms of spectral theory of self-adjoint transformations in the separable Hilbert space: [5], a resolution of identity $\{P_t, t \geq 0\}$ in Hilbert space $\mathcal{H}(X)$ is defined by $P_t \mathcal{H}(X) = \mathcal{H}(X,t)$. The relation (7) means that the space $\mathcal{H}(X)$ is subspace of $\mathcal{H}(W)$ reducing $\{P_t, t \geq 0\}$. One can pass from canonical representation $X = [g, W]$ to proper canonical one by the suitable choice of wide-sense martingale $\{Z(t), t \geq 0\}$, $\mathcal{H}(Z,t) = \mathcal{H}(X,t)$, with the structural function $F_Z(du) = \phi(u)du$, $\phi(u) \geq 0$ (a.e. du):

$$X(t) = \int_0^t g(t,u) Z(du) = \int_0^t g(t,u) \sqrt{\phi(u)} W(du). \tag{8}$$

Then the representation (8) is proper canonical: $X = [g, Z] = [g\sqrt{\phi}, W]$.

Proposition 5. *If (6) is the proper canonical representation of* $X = \{X(t), t > 0\}$ *and the derivative* $\dot{X} = \{\dot{X}(t), t > 0\}$ *exists, then*

$$\dot{X}(t) = \int_0^t f(t,u)W(du), \quad \|\dot{X}(t)\|^2 = \int_0^t f^2(t,u)du \tag{9}$$

is the proper canonical representation of $\dot{X}$.

Proof. We will use the following characterization [4]: the representation (6) is proper canonical if and only if from

$$\int_0^t g(t,u)h(u)du = 0, \quad \forall t \geq 0 \tag{$*$}$$

it follows that $h(u) = 0$ (a.e. du). Therefore, we only need to show that from

$$\int_0^t f(t,u)h(u)du = 0, \quad \forall t \geq 0 \tag{$**$}$$

the relation $(*)$ follows, for any $h \in L_2(du)$. But, if $(**)$ holds we have

$$\begin{aligned}\int_0^t g(t,u)h(u)du &= \int_0^t \left\{\int_u^t f(s,u)ds\right\} h(u)du \\ &= \int_0^t \left\{\int_0^s f(s,u)\right\} h(u)duds = 0\end{aligned}$$

so $h(u) = 0$ (a.e. du) and the representation (9) is proper canonical. $\square$

Remark 2. The concept of proper canonical representation was generalized in Cramèr's paper [1] as the theory of spectral multiplicity of second order processes. From this theory it follows that for any fixed integer M, finite or infinite, there exists mean-square continuous process $\{X_*(t), t \geq 0\}$ with the proper canonical representation

$$X_*(t) = \sum_{n=1}^M \int_0^t g_n(t,u)W_n(du), \quad \|X_*(t)\|^2 = \sum_{n=1}^M \int_0^t g_n^2(t,u)du \tag{10}$$

where:

1^0 $\{W_n(t), t \geq 0\}, n = 1, \ldots, M$ are mutually orthogonal wide-sense Wiener processes;

2^0 $W_n(t) \in \mathcal{H}(X_*; t), t \geq 0, n = 1, 2, \ldots, M$;

3^0 $\mathcal{H}(X_*; t) = \sum_{n=1}^M \mathcal{H}(W_n; t), t \geq 0$.

We use asterix $*$ in the notation X_* to refer to the process $X_* = \{X_*(t), t \geq 0\}$ of the spectral multiplicity $M \geq 2$. It was proved in [2] that (10) is the proper canonical representation if and only if from

$$\sum_{n=1}^{M} \int_0^t h_n(u) g_n(t,u)du = 0, \quad \forall t \geq 0$$

it follows $h_n(t) = 0$ (a.e. dt), $n = 1, \dots, M$.

Proposition 6. *The process X_* has a mean-square derivative $\dot{X}_*$ if and only if $g_n(t,u) = \int_u^t f_n(x,u)dx$ for $n = 1, \dots, M$, and*

$$\dot{X}_*(t) = \sum_{n=1}^{M} \int_0^t f_n(t,u) W_n(du) \tag{11}$$

provided that the series (11) *is uniformly convergent. The representation* (11) *is proper canonical representation of $\dot{X}_*$.*

We omit the easy proof.

Example 3. We give the example of mean-square derivative $\dot{X}_*$ of the process X_* with the spectral multiplicity $M \geq 2$. Suppose $t \in [0,1]$, and let $A_1, \dots, A_M$ be mutually disjoint subsets of $[0,1]$, such that $\bigcup_{n=1}^{M} A_n = [0,1]$ and for any nonempty $(a,b) \subset [0,1]$ the Lebesgue measure of $A_n \cap (a,b)$ is positive for each $n = 1, \dots, M$, (see [1]). Denote by $\chi_n(t)$ the indicator function of A_n. Let

$$g_n(t,u) = \int_u^t \chi_n(v)dv.$$

Then $f_n(t,u) = \chi_n(t)$ and

$$\dot{X}_*(t) = \sum_{n=1}^{M} \int_0^t \chi_n(t) W_n(du) = \sum_{n=1}^{M} \chi_n(t) W_n(t).$$

3. Hida in [4] develops the concept of $\mathbb{N}$-ple Gaussian Markov processes as the processes having proper canonical Goursat kernel

$$g(t,u) = \sum_{k=0}^{N} f_k(t) g_k(u).$$

We consider a particular case of Goursat kernel and show that it is proper canonical kernel.

Proposition 7. *Let*

$$g(t,u)=\sum_{k=0}^{N} t^k \alpha_k(u), \quad (\alpha_N(u)=1)$$

be the kernel of a non-anticipative transformation (6) *and let* $\dot{X}^{(N)}(t)$ *be different from zero. Then*

$$g(t,u)=(t-u)^N$$

and the representation is proper canonical.

(The assumption $\alpha_N(u)=1$ is not essential.)

PROOF. Rewriting the kernel $g(t,u)$ as Taylor polynomial in t at the point s we get

$$\begin{aligned} g(t,u) &= g(s,u)+\frac{(t-s)}{1!}\,\frac{\partial}{\partial t}g(t,u)|_{t=s} \\ &\quad +\cdots+\frac{(t-s)^{N-1}}{(N-1)!}\,\frac{\partial^{N-1}}{\partial t^{N-1}}g(t,u)|_{t=s}+(t-s)^N \end{aligned} \tag{9}$$

$\left(\frac{\partial^N}{\partial t^N}g(t,u)=N!\right)$. Putting $u=s$ and using existence of derivatives $\dot{X}^{(k)}(t), 1\le k\le N$, we obtain $g(t,u)=(t-s)^N$.

The correlation function of $\{X(t)\}$ is

$$r(t,s)=\big\langle X(t),X(s)\big\rangle=\int_0^s (t-u)^N(s-u)^N du, \quad t\ge s$$

We have, for $t>s\ge v$

$$r(t,v)=r(s,v)+\frac{(t-s)}{1!}\,\frac{\partial}{\partial t}r(t,v)|_{t=s}+\cdots+\frac{(t-s)^N}{N!}\,\frac{\partial^N}{\partial t^N}r(t,v)|_{t=s}$$

and

$$\begin{aligned}\big\langle X(t),X(v)\big\rangle &= \big\langle X(s),X(v)\big\rangle+\frac{(t-s)}{1!}\big\langle \dot{X}(s),X(v)\big\rangle \\ &\quad +\cdots+\frac{(t-s)^N}{N!}\big\langle \dot{X}^{(N)}(s),X(v)\big\rangle,\end{aligned}$$

so

$$\Big\langle X(t)-\sum_{k=0}^{N}\frac{(t-s)^k}{k!}\dot{X}^{(k)}(s),X(v)\Big\rangle=0$$

for all $v \leq s$. The last equality shows that the linear prediction $\overline{X}(t,s) = P_s X(t)$ is

$$\overline{X}(t,s) = \sum_{k=0}^{N} \frac{(t-s)^k}{k!} \dot{X}^{(k)}(s)$$

(P_s is the projection operator on $\mathcal{H}(X;s)$. Taking $\int_0^s (\,.\,)W(du)$ on the both sides in (9), we obtain

$$\int_0^s g(t,u)W(du) = \sum_{k=0}^{N} \frac{(t-s)^k}{k!} \dot{X}^{(k)}(s)$$

or

$$\int_0^s (t-u)^N W(du) = \overline{X}(t,s),$$

so we conclude that the representation

$$X(t) = \int_0^t (t-u)^N W(du)$$

is canonical. It is in fact proper canonical, as it follows from the remark above. $\square$

Using previous results, we can estimate the linear prediction $\overline{Y}(t;s)$ and a Taylor polynomial $\tilde{Y}(t;s)$ and find the error of this estimation.

Proposition 8. *Let $\{Y(t)\}$ have mean-square continuous $(N+1)$-th derivative on $[s,t]$ for some $N = 0,1,2,\ldots$ and let*

$$c = \max_{s \leq v \leq t} \|Y^{(N+1)}(v)\|^2.$$

Then the linear prediction $\overline{Y}(t;s)$ can be approximated by

$$\tilde{Y}(t;s) = \sum_{k=0}^{N} \frac{(t-s)^k}{k!} \dot{Y}^{(k)}(s)$$

with mean-square error of this approximation

$$\|\overline{Y} - \tilde{Y}\|^2 \leq c\Big[\frac{(t-s)^{N+1}}{(N+1)!}\Big]^2.$$

Proof. For $u \leq s < \xi < t$ the expansion

$$r(t,u) = r(s,u) + \frac{(t-s)}{1!}\frac{\partial}{\partial t} r(t,u)|_{t=s} + \cdots + \frac{(t-s)^N}{N!}\frac{\partial^N}{\partial t^N} r(t,u)|_{t=s}$$
$$+ \frac{(t-s)^{N+1}}{(N+1)!}\frac{\partial^{N+1}}{\partial t^{N+1}} r(t,u)|_{t=\xi},$$

yields, for all $u \leq s$

$$\left\langle Y(t) - \tilde{Y} - \frac{(t-s)^{N+1}}{(N+1)!} \dot{Y}^{(N+1)}(\xi), Y(u) \right\rangle = 0$$

or

$$P_s\big(Y(t) - \frac{(t-s)^{N+1}}{(N+1)!} \dot{Y}^{(N+1)}(\xi)\big) = \tilde{Y}$$

so it follows that

$$\overline{Y} - \tilde{Y} = P_s \left(\frac{(t-s)^{N+1}}{(N+1)!} \dot{Y}^{(N+1)}(\xi) \right).$$

Finally,

$$\|\overline{Y} - \tilde{Y}\|^2 \leq \left\| \frac{(t-s)^{N+1}}{(N+1)!} \dot{Y}^{(N+1)}(\xi) \right\|^2 \leq c \left[\frac{(t-s)^{N+1}}{(N+1)!} \right]^2. \qquad \square$$

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