Multiplicative congruences on matrixsemigroups

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In memorian of Andor Kertész

Let R be a ring with unity and $M_n(R), n>1$, the matrixring with unity E over the ring R. The group of units of R and $M_n(R)$ will be denoted by $G(R)$ and $GL_n(R)$ and the elementary matrices of $M_n(R)$ by $D_q(a)$, $T_{pq}(b)$ and P_{pq}; $p, q \in \{1, 2, \dots, n\}$. The matrix E_{pq} represents a matrix with 1 as (p, q)-component and every other component zero.

Let $\mathscr{C}$ be the set of all multiplicative semigroup congruences δ on $M_n(R)$ such that the factorsemigroup $M_n(R)/\delta$ is commutative and $\mathscr{D}$ the set of all multiplicative semigroup congruences σ on $M_n(R)$ such that $T_{pq}(b) \sim E(\sigma)$ for all $p, q \in \{1, 2, \dots$ $\dots, n\}$ and for all b of R.

Lemma 1. *$T_{pq}(b)$ is a groupcommutator of $GL_n(R)$ for all $b \in R$ and for $p \notin \{q-1, q+1\}$.*

Proof. It can be checked that for $p \in \{q+2, q+3, \dots\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

in which

$$X = T_{p-1,q}(-b) + E_{p,p-1}$$

$$Y = T_{p-1,q}(b)$$

and for $p \in \{\dots, q-3, q-2\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

where

$$X = T_{p+1,q}(-b) + E_{p,p+1}$$

$$Y = T_{p+1,q}(b)$$

Lemma 2. *Let R be a ring with an element a such that $\{a, (1-a)\} \subseteq G(R)$ then $T_{q-1,q}$ and $T_{q+1,q}$ are groupcommutators in $GL_n(R)$*

Proof. It can be verified that for $p \in \{q-1, q+1\}$

$$T_{pq}(b) = XYX^{-1}Y^{-1}$$

where

$$X = T_{pq}((1-a)^{-1} \cdot b)$$
$$Y = D_p(a)$$

From lemma 1 and lemma 2 one deduce immediately the following

Theorem 1. *If R is a ring containing an element a such that $\{a, (1-a)\} \subseteqq G(R)$, then $\mathscr{C} \subseteqq \mathscr{D}$.*

In the following we consider only rings R such that the multiplicative semigroup R and the matrixsemigroup $M_n^\circ(R)$ are semigroups with separating group part [1], p. 256. We denote by

— $D_q(R)$, D, P and $TL_n(R)$ respectively the sets $\{D_q(r) \| r \in R\}$, $\cup D_q(R)$, $\{P_{pq} \| p, q \in \{1, 2 \dots, n\}$ and $\{T_{pq}(r) \| r \in R$ and $p, q \in \{1, 2, \dots, n\}\}$
— $M_n^\circ(R)$ the multiplicative subsemigroup of $M_n(R)$ generated by the set $D \cup P \cup \cup TL_n(R)$
— $GL_n^\circ(R)$ the group of units of $M_n^\circ(R)$ with commutatorgroup $CL_n^\circ(R)$.
— Φ the empty set.

Lemma 3. *$TL_n(R)$ is an admissible set,* [2] p 178] *in $M_n^\circ(R)$ and an invariant subgroup of $GL_n^\circ(R)$ containing the commutatorgroup $CL_n^\circ(R)$ of $GL_n^\circ(R)$.*

Proof. We have to prove that for all matrices A of $GL_n^\circ(R)$

(1) $$A TL_n(R) = TL_n(R) A$$

Since

$$D_q(r) = P_{1q} \cdot D_1(r) \cdot P_{1q}$$

and

$$P_{pq} = T_{pq}(1) \cdot T_{qp}(-1) \cdot T_{pq}(1) \cdot D_p(-1)$$

(1) reduces to
(2) for all r belonging to $G(R)$: $D_1(r) TL_n(R) = TL_n(R) D_1(r)$ which follows from the identities

(3) $$D_p(a) \cdot T_{rs}(b) = T_{rs}(b) \cdot D_p(a) \quad \text{if} \quad p \neq r \quad \text{and} \quad p \neq s$$
$$D_p(a) \cdot T_{ps}(b) = T_{ps}(a \cdot b) \cdot D_p(a) \quad \text{if} \quad p \neq s$$
$$D_p(a) \cdot T_{rp}(b) = T_{rp}(b \cdot a^{-1}) \cdot D_p(a) \quad \text{if} \quad r \neq p$$

For all $p, q \in \{1, 2, \dots, n\}$ and for all $a, b \in G(R)$ we have

$$P_{pq} TL_n(R) = D_p(-1) TL_n(R)$$
$$D_p(a) TL_n(R) = D_q(a) TL_n(R)$$
$$D_p(a \cdot b) TL_n(R) = D_p(b \cdot a) TL_n(R)$$

and hence, for all $A, B \in GL_n^\circ(R)$:

$$ABTL_n(R) = BATL_n(R)$$

Suppose that

$$ATL_n(R) B \cap TL_n(R) \neq \emptyset \quad \text{with} \quad A, B \in M_n^\circ(R)$$

or, equivalently

$$ATL_n(R)\,B \cap TL_n(R) \neq \emptyset \quad \text{with} \quad A, B \in GL_n^\circ(R)$$

or,

$$ABTL_n(R) \cap TL_n(R) \neq \emptyset \quad \text{with} \quad A, B \in GL_n^\circ(R)$$

Then

$$ABTL_n(R) \subseteqq TL_n(R)$$

or

$$ATL_n(R)\,B \subseteqq TL_n(R),$$

from which we may conclude that $TL_n(R)$ is an admissible set in the semigroup $M_n^\circ(R)$.

It is well known, see [2], p. 179, that every semigroupcongruence δ on the multiplicative semigroup $M_n^\circ(R)$ which admits $TL_n(R)$ satisfies the relation

$$\alpha C^* T \subseteqq \delta \subseteqq \beta C \tag{4}$$

in which

$$\alpha = \big(TL_n(R) \times TL_n(R)\big) \cup \{(A, A) \,\|\, A \in M_n^\circ(R) \setminus TL_n(R)\}$$

$$\beta = \big(TL_n(R) \times TL_n(R)\big) \cup \big(M_n^\circ(R) \setminus TL_n(R) \times M_n^\circ(R) \setminus TL_n(R)\big)$$

and

$\alpha C^* T =$ the unique minimal congruence containing α

$\beta C =$ the unique maximal congruence contained in β.

Definition. A semigroupconvergence δ satisfying the relation (4) will be called a *T-congruence* on the multiplicative semigroup $M_n^\circ(R)$.

Theorem 2. *If δ is a T-congruence on the multiplicative semigroup $M_n^\circ(R)$ then the factorsemigroup $M_n^\circ(R)/\delta$ is commutative and there exists a semigroup-congruence ϱ on the multiplicative semigroup R such that $M_n^\circ(R)/\delta$ is isomorphic with R/ϱ.*

Proof. If $\bar{A}$ and $\bar{B}$ represent any two elements of $M_n^\circ(R)/\delta$ then $\bar{A}$ contains a matrix of the form $D_1(a)$ and $\bar{B}$ contains a matrix of the form $D_n(b)$ for which

$$\overline{D_1(a)} \cdot \overline{D_n(b)} = \overline{D_n(b)} \cdot \overline{D_1(a)}$$

and hence

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}.$$

The restriction of the semigroupcongruence δ to the subsemigroup $D_1(R)$ of $M_n^\circ(R)$ induces a congruence ϱ on the multiplicative semigroup R of the ring R since the multiplicative semigroup R is isomorphic with $D_1(R)$. The stated semigroupisomorphism follows from the fact that each class of $M_n^\circ(R)/\delta$ contains elements of $D_1(R)$.

If $M_n^\circ(R) \neq M_n(R)$, then one can introduce the following equivalence relation σ on $M_n(R)$:

$$X \sim Y(\sigma) \Leftrightarrow \sum_{\pi} \operatorname{sgn} \pi\, x_{1,\pi(1)}, x_{2,\pi(2)}, \ldots, x_{n,\pi(n)} \sim$$

$$\sim \sum_{\pi'} \operatorname{sgn} \pi'\, y_{1,\pi'(1)}, y_{2,\pi'(2)}, \ldots, y_{n,\pi'(n)}(\varrho) \ldots$$

in which π and π' are permutations on the set $\{1, 2, \ldots, n\}$ and ϱ the semigroupcongruence on R of theorem 2.

Theorem 3. *If the associated semigroupcongruence ϱ on R of a T-congruence δ on $M_n^\circ(R)$ is also a ringcongruence on R, then the relation σ on $M_n(R)$ is a semigroupcongruence.*

PROOF. The result follows from the fact that the factorring R/ϱ is commutative.

Examples

1. *Matrixrings over division rings.*

The multiplicative semigroup $M_n^\circ(L)$ over a division ring L is a semigroup with separating group part. Moreover $M_n(L)=M_n^\circ(L)$ and by theorem 1, the set $TL_n(L)$ is an invariant subgroup of the group $GL_n(L)$ containing the commutatorgroup $CL_n(L)$ of $GL_n(L)$ which is a well-known result since for $n\neq 2$ and for $n=2$ with $L\neq\{0, 1\}$ one has $TL_n(L)=CL_n(L)$. For $n=2$ together with $L=\{0, 1\}$ one has $CL_2(\{0, 1\})\subsetneqq TL_2(\{0, 1\})$. Moreover, there exists a semigroupcongruence δ on $M_n^\circ(L)$ which admits $TL_n(L)$. By theorem 2, there exists a nontrivial semigroupcongruence ϱ on the division ring L such that $M_n(L)/\delta$ is semigroupisomorphic with L/ϱ. Since L/ϱ is a group with zero, the factorsemigroup $M_n(L)/\delta$ will be also a group with zero and with unity $TL_n(L)$. By the results of P. Dubreil, [2], p. 182, δ must be unique and be equal to

$$R_{TL_n(L)} = \{(A, B)\in \operatorname{cart}^2 M_n(L) \,\|\, AX\in TL_n(L) \Leftrightarrow BX\in TL_n(L), \quad \forall X\in M_n(L)\}$$

It is clear that, unless $n=2$ together with $L=\{0, 1\}$, δ will be the semigroupcongruence determined by the determinantmorphism defined by J. Dieudonné [3], for which $GL_n(L)/CL_n(L)$ is groupisomorphic with $G(L)/C$, in which $CL_n(L)$ represents the commutatorgroup of the group $GL_n(L)$ and C the commutatorgroup of the group $G(L)$. In the case $n=2$ together with $L=\{0, 1\}$ it can be proved that $M_2(\{0, 1\})/\delta$ is semigroupisomorphic with the group with zero $\{0, 1, x \,\|\, x^2=1\}$.

2. *Matrixrings over commutative rings.*

Let R_c be any commutative ring. The multiplicative semigroups R_c and $M_n^\circ(R_c)$ are semigroups with separating group part. By theorem 1, the subset $TL_n(R_c)$ of the matrixring $M_n(R_c)$ over a commutative ring R_c is an invariant subgroup of the group $GL_n^\circ(R_c)$ containing the commutatorgroup $CL_n^\circ(R_c)$ of $GL_n^\circ(R_c)$ and $TL_n(R_c)$ is an admissible set in $M_n^\circ(R_c)$. Every T-convergence δ admitting $TL_n(R_c)$ satisfies the relation $\alpha C^* T\subseteq\delta\subseteq\beta C$. The T-congruences $\alpha C^* T$ and βC on $M_n^\circ(R_c)$ can be characterised as follows.

Let $\delta\mathrm{et}_r$ denote the semigroupcongruence associated with the restriction $\det_r$ of the determinant on $M_n^\circ(R_c)$.

Proposition 1: $\alpha C^* T=\delta\mathrm{et}_r$

PROOF. Clearly $TL_n(R_c)\subseteq \det_r^*\{1\}=\{M\in M_n^\circ(R_c) \,\|\, \det M=1\}$ Conversely, let A be a matrix of $M_n^\circ(R_c)$ such that $\det_r A=1$. From (3) it follows that A can be written as a product of $T_{pq}(b)$-matrices and a diagonalmatrix diag $(a_{11}, a_{22}, \ldots, a_{nn})$

with $\prod_{i=1}^{n} a_{ii}=1$. This diagonalmatrix will be called an associated diagonalmatrix D_A of the matrix A of $M_n^\circ(R_c)$. This matrix D_A also belongs to $TL_n(R_c)$ since D_A can be written as a product of matrices of the form $D_i(r)\cdot D_j(r^{-1})$ for which $D_i(r)\cdot \cdot D_j(r^{-1})=T_{ij}(r)\cdot T_{ji}(1-r^{-1})\cdot T_{ij}(-1)\cdot T_{ji}(1-r)$ Therefore A belongs to $TL_n(R_c)$. This means that $\alpha C^* T \subseteq \delta\mathrm{et}_r \subseteq \beta C^* T$ and it remains to prove that $\delta\mathrm{et}_r \subseteq \alpha C^* T$. Let $(A, B)\in \mathrm{cart}^2\, M_n^\circ(R_c)$, then from $A\sim B(\delta\mathrm{et}_r)$ it follows that $\det_r A=\det_r B$ and $D_1(\det_r A)=D_1(\det_r B)$. If D_A and D_B represent respectively associated diagonalmatrices of the matrices A and B then clearly $D_A\sim A\,(\alpha C^* T)$ and $D_B\sim B\,(\alpha C^* T)$. Since also $D_A\sim D_1(\det_r A)\,(\alpha C^* T)$ and $D_B\sim D_1(\det_r B)\,(\alpha C^* T)$ we obtain $D_A\sim D_B(\alpha C^* T)$ and finally $A\sim B(\alpha C^* T)$.

Proposition 2. *If R_c is a commutative ring then the unique maximal congruence admitting $TL_n(R_c)$ of $M_n(R_c)$ equals the Dubreilcongruence associated with the set $TL_n(R_c)$.*

Proof. Since $TL_n(R_c)$ is a normal divisor of $GL_n^\circ(R_c)$ and co $\left(GL_n^\circ(R_c)\right)$ a primesemigroupideal of $M_n^\circ(R_c)$, the partition determined by the cosets of $TL_n(R_c)$ in $GL_n^\circ(R_c)$ and the set co $\left(GL_n^\circ(R_c)\right)$ determines a congruence δ on $M_n^\circ(R_c)$ such that $M_n^\circ(R_c)/\delta$ will be a group with zero. Therefore δ must be the Dubreil-congruence associated with the set $TL_n(R_c)$. It is the unique maximal congruence admitting $TL_n(R_c)$ since any congruence which is greater than δ will be the trivial congruence on $M_n^\circ(R_c)$.

Corollary 1. Any T-congruence on $M_n^\circ(R_c)$ is greater than the determinant-congruence $\delta\mathrm{et}_r$ and smaller than the Dubreil-congruence associated to the set $TL_n(R_c)$ in $M_n^\circ(R_c)$.

Corollary 2. If R_E denotes an Euclidean ring then $M_n^\circ(R_E)=M_n(R_E)$ and therefore one has : any T-convergence on $M_n(R_E)$ is greater than the determinant-congruence $\delta\mathrm{et}$ and smaller than the Dubreil-congruence associated to the set $TL_n(R_c)$ in $M_n(R_E)$.

The multiplicative congruence ϱ on R corresponding with the multiplicative congruence $\delta\mathrm{et}_r$ on $M_n^\circ(R_c)$ consist of the diagonal Δ of $\mathrm{cart}^2\, R$ and therefore is a ringcongruence on R_c. By theorem 3 the relation

$$X\sim Y(\sigma) \Leftrightarrow \sum_{\pi} \mathrm{sgn}\,\pi\, x_{1,\pi(1)},\, x_{2,\pi(2)},\, \ldots,\, x_{n,\pi(n)} =$$

$$= \sum_{\pi'} \mathrm{sgn}\,\pi'\, x_{1,\pi'(1)}\cdot x_{2,\pi'(2)},\, \ldots,\, x_{n,\pi'(n)}$$

must be a multiplicative congruence on $M_n(R_c)$ which is indeed the case since it is the multiplicative congruence associated with the determinant on $M_n(R_c)$.

References

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