Full modules over semifirs

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In memoriam Andor Kertész

1. Let R be a principal ideal domain (not necessarily commutative), then every finitely generated R-module M has a presentation

$$0 \to R^n \to R^m \to M \to 0. \tag{1}$$

Here $n \leqq m$, with equality if and only if M is a torsion module. All this holds more generally for any finitely presented module over a Bezout domain, i.e. an integral domain in which finitely generated left or right ideals are principal.

By a *right fir* (=free ideal ring) we understand a ring in which all right ideals are free, of unique rank; *left firs* are defined similarly, and a *fir* is a left and right fir. Firs form a natural extension of the class of principal ideal domains, in the sense that many properties of principal ideal domains can be generalized to firs; thus firs themselves in the commutative case are just principal ideal domains. In particular it is possible to develop a theory of torsion modules over firs, which generalizes the usual notion of torsion module over a principal ideal domain (cf. [1, 2, 3, 4]). The corresponding generalization of a Bezout domain is a *semifir*, i.e. a ring in which every finitely generated right ideal is free, of unique rank; this notion turns out to be left-right symmetric (in contrast to firs). Much of the theory of torsion modules actually applies to semifirs, when restricted to finitely presented modules.

Our object in this note is to examine the class of all finitely presented modules over a semifir. Of course we cannot expect the same neat classification as for modules over a principal ideal domain, where every finitely generated module is a direct sum of cyclic modules, but we shall find a subclass, the *full* modules, which shows good behaviour and from which the other modules can be built up.

2. Throughout, all rings are associative, with a unit-element 1, which is inherited by subrings, preserved by homomorphisms and acts unitally on modules. The modules are usually *right* modules when nothing is said to the contrary. If R is a ring, the set of all $m \times n$ matrices over R is denoted by ${}^mR^n$; any exponent equal to 1 is often omitted, thus we write ${}^mR$ for ${}^mR^1$ (the set of columns of length m) and R^n for ${}^1R^n$ (the set of rows of length n).

Let R be any ring and $A \in {}^mR^n$, then the least integer r such that $A = PQ$, where P is $m \times r$ and Q is $r \times n$ is called the *inner rank* of A. An $n \times n$ matrix is said to be *full* if its inner rank is n. Over a semifir, the inner rank of a matrix A

is the largest order of any full submatrix formed from the rows and columns of A (cf. [5]).

Given a ring R, we recall that any $A \in {}^mR^n$ gives rise to a finitely presented R-module M, obtained as the cokernel of the mapping α defined by A:

(2) $$ {}^nR \xrightarrow{\alpha} {}^mR \to M \to 0. $$

We note that $M=0$ if and only if $AB=I$ for some $B \in {}^nR^m$. When α in (2) is injective, i.e. when A is a left non-zerodivisor, we can write the presentation of M as

(3) $$ 0 \to {}^nR \xrightarrow{\alpha} {}^mR \to M \to 0. $$

In this case we shall call A a *presenting matrix* for M. In particular, when R is a semifir, any finitely presented R-module M has a presentation (3), for the image of α is then a finitely generated submodule of a free module, and this is necessarily free. In that case we call $m-n$ the *characteristic* of M and write $\chi(M)=$ $=m-n$. An application of Schanuel's lemma shows that $\chi(M)$ does not depend on the presentation used.

3. Given a module M over a semifir R, with presentation (3), let us dualize and write $M^*=\mathrm{Hom}\,(M, R)$ for short. We obtain the exact sequence

(4) $$ 0 \to M^* \to R^m \to R^n \to \mathrm{Ext}^1_R(M, R) \to 0. $$

This shows that A is a right non-zerodivisor precisely when $M^*=0$, i.e. M is *bound* in the terminology of [3].

Definition. A module M over a semifir R is said to be *full* if it is bound, with a presentation (3) such that the presenting matrix has inner rank $\min\,\{m, n\}$.

A full module will show different behaviour according to the sign of its characteristic. Let us call an R-module M *positive*, *negative* or *torsion* if M is full and $\chi(M)$ is $\geqq 0$, $\leqq 0$ or 0 respectively. This agrees with the definition of torsion module given in Ch. 5 of [3]. The corresponding classes of right R-modules are denoted by Pos_R, Neg_R, Tor_R and the left modules by ${}_R\mathrm{Pos}$, ${}_R\mathrm{Neg}$ and ${}_R\mathrm{Tor}$. Clearly we have

(5) $$ \mathrm{Tor}_R = \mathrm{Pos}_R \cap \mathrm{Neg}_R. $$

Moreover, the duality for finitely presented bound modules of projective dimension at most 1 ([3], Prop. 5.2.1) shows the truth of

Theorem 1. *Let R be a semifir, then the functor $\mathrm{Ext}^1_R(-, R)$ provides a duality between the class of finitely presented bound right R-modules and the corresponding class of left R-modules, under which Pos_R corresponds to ${}_R\mathrm{Neg}$.*

Over a principal ideal domain (commutative or not) or even a Bezout domain, every full module is positive, essentially because then any finitely generated submodule of ${}^nR$ has rank at most n, and so $\chi(M) \geqq 0$. But in general it is easy to construct full modules of arbitrary negative characteristic. E.g. the free algebra $F=k\langle x, y\rangle$ is a fir, and the elements $x^i y$ $(i=0, 1, \ldots, n)$ freely generate a right ideal $\mathfrak{a}$ of rank $n+1$, hence $\chi(F/\mathfrak{a})=-n$.

4. Consider a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0. \tag{6}$$

If M can be generated by m elements, then so can M'', and resolving M, M'' in a corresponding way we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & {}^nR & \to & {}^nR & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & {}^rR & \to & {}^mR & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{7}$$

with exact rows and columns. Here we can always arrange matters so that the map in the top row is the identity. If the presenting matrices of M', M, M'' are C, A, B respectively, then we have

$$A = BC, \tag{8}$$

and it is easy to establish the additivity of the characteristic from the diagram (7) (cf. [3]):

$$\chi(M) = \chi(M') + \chi(M'').$$

Conversely, any equation (8) between matrices which are left non-zerodivisors can be realized by a diagram (7) giving rise to a short exact sequence (6). This leads to the following invariant description of full modules:

Theorem 2. *Let R be a semifir and M a finitely presented R-module, then* (i) *M is positive if and only if M is bound and $\chi(M') \geqq 0$ for all submodules M' of M,* (ii) *M is negative if and only if $\chi(M'') \leqq 0$ for all quotients M'' of M.*

For the proof we need only observe that any module satisfying (ii) is necessarily bound: if $M^* \neq 0$, then by Prop. 5.1.1 of [3], $M = N \oplus R$, hence R is a quotient of M, but $\chi(R) = 1 > 0$.

The next result tells us when submodules or extensions of full modules are full. Since every full module is either positive or negative, it is convenient to treat these cases separately:

Theorem 3. *Let R be a semifir and* (6) *a short exact sequence of R-modules. If M', M'' are both positive, then so is M; conversely, if M is positive, so is every bound submodule, but not every quotient. If M', M'' are both negative, then so is M; conversely, if M is negative, then so is every quotient, but not necessarily every submodule.*

Proof. From (6) we obtain the exact sequence

$$0 \to M''^* \to M^* \to M'^*,$$

hence if M', M'' are bound, then so is M. Assume further that M', M'' are positive, and let $N \subseteq M$. Then $\chi(N \cap M') \geqq 0$ and

$$N/(N \cap M') \cong (N+M')/M' \subseteq M/M' \cong M'',$$

hence $\chi(N/(N \cap M')) \geqq 0$, therefore $\chi(N) = \chi(N/(N \cap M')) + \chi(N \cap M') \geqq 0$, and this shows M to be positive. If M is positive, then any submodule N of M' satisfies $\chi(N) \geqq 0$, hence any bound submodule M' of M is positive.

However, M'' need not be positive, e.g. if $\chi(M)=0$, $\chi(M')>0$, then $\chi(M'')<0$. Likewise not every submodule of a positive module is bound. As an example for both these cases we can again take $F=k<x, y>$; here $M=F/xF$ is torsion, but it has the submodule $(xF+yF)/xF \cong yF/(xF \cap yF) \cong F$, with quotient $F/(xF+yF)$ of characteristic -1.

The corresponding result for negative modules follows by duality.

Corollary 1. *In any finitely presented module M over a semifir, the sum N of all negative submodules is itself negative, and the quotient M/N has no non-zero negative submodules. Moreover, N is the submodule for which $\chi(N)$ is least and it is the largest submodule subject to this condition.*

PROOF. Let N_1, N_2 be negative submodules of M, then N_1+N_2, as homomorphic image of $N_1 \oplus N_2$ is again negative. Now any submodule M' of a module M with presentation (3) satisfies

$$\chi(M') \geqq -n, \tag{9}$$

hence the characteristics of submodules of M are bounded below. Let N be the union of all the negative submodules of M, then by (9) N must be finitely related, and hence it is the direct sum of a free module and a finitely presented module. But any finitely generated submodule of N is negative, so the free component is zero, N is itself finitely generated and hence negative. If M/N had a non-zero negative submodule M'/N, then M' as an extension of negative modules would be negative, but this contradicts the choice of N. Hence M/N has no non-zero negative submodules, as claimed.

Finally let N' be a submodule of M such that $\chi(N')$ is least, then any submodule N_1 of N' satisfies $\chi(N_1) \geqq \chi(N')$, hence $\chi(N'/N_1) \leqq 0$, therefore N' is negative and so $N' \subseteq N$. Moreover, $\chi(N') \leqq \chi(N)$ by the choice of N', and here equality holds because N is negative. Thus N is indeed the largest submodule with the least characteristic in M. This completes the proof.

Of course an extension of full modules need not be full, in fact we shall soon see that every bound module can be written as an extension of full modules.

From the definitions we have the

Corollary 2. *Let $f: M \to N$ be a homomorphism, where M, N are negative modules over a semifir, then coker f is also negative.*

The dual of this result is false: the kernel of a mapping between positive modules need not be positive (because it need not be bound). E.g. let A be a full $n \times n$ matrix and A' the $n \times (n-1)$ matrix consisting of the first $n-1$ columns of A, then $A' = A\begin{pmatrix} I \\ 0 \end{pmatrix}$, and if (6) is the exact sequence corresponding to this factorization, then M is the positive module defined by A', M'' is the torsion module defined by A and M' is defined by $\begin{pmatrix} I \\ 0 \end{pmatrix}$; the latter is clearly not bound. Of course the kernel of a mapping between negative modules need not be negative (Th. 3), nor the cokernel of a mapping between positive modules positive. Neither need an endomorphism of a negative module have a negative kernel, as the following remark of G. M. Bergman shows: let $f: M \to N$ be any homomorphism between negative modules whose kernel is not negative and consider the endomorphism $\begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$ of $M \oplus N$.

5. Finally we have the following description of arbitrary finitely presented modules in terms of full modules. We first make a reduction to bound modules.

Theorem 4. *Let M be a finitely generated module over a semifir R, then $M = N \oplus {}^nR$, where N is a bound module.*

Proof. Let M be generated by m elements, then every homomorphic image of M can be generated by m elements. If M is not bound, then $M^* = \mathrm{Hom} \cdot (M, R) \neq 0$. The image of M in R under a homomorphism is a finitely generated right ideal of R, hence free, and so a direct summand of M. Thus we can write

$$M = N \oplus {}^nR, \tag{10}$$

where $n > 0$, unless M is bound. Moreover, $n \leqq m$, for ${}^nR$ is a homomorphic image of M and cannot be genereted by fewer than n elements (because R is ‚weakly finite', cf. [3]). If we choose n as large as possible in (10), then N does not have R as a direct summand, and hence is bound. This completes the proof.

Next we express bound modules in terms of full modules.

Theorem 5. *Let R be a semifir and M any finitely presented bound R-module. If $\chi(M) \geqq 0$, then M has a unique greatest negative submodule with positive quotient; if $\chi(M) \leqq 0$, M has a unique least negative submodule with positive quotient.*

Proof. Suppose first that $\chi(M) \geqq 0$ and let N be the maximal negative submodule which exists by Th. 3, Cor.1. Put $P = M/N$, then $\chi(P) = \chi(M) - \chi(N) \geqq 0$. By the description of N, $\chi(M') \geqq \chi(N)$ for any submodule M' of M, hence if $M' \supseteqq N$, then $\chi(M'/N) \geqq 0$, i.e. all submodules of P have positive characteristic. Moreover, P is bound, as homomorphic image of a bound module, therefore P is positive.

Now the second assertion follows by duality, using Th .1.

Combining Th. 4 and 5, we obtain the

Corollary. Every finitely presented module over a semifir is a direct sum of a free module and an extension of a negative by a positive module.

References

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