A note on semisimple classes

By PAUL O. ENERSEN (Marshall, Minnesota)
and W. G. LEAVITT (Lincoln, Nebraska)

Dedicated to the memory of Andor Kertész

1. *Introduction.*

For a given (not necessarily associative) ring R, we will let $\bar{R}$ denote any non-zero homomorphic image of R, and write $I \lhd R$ when I is a non-zero ideal of R. For a given class M we have as usual the dual definitions $UM = \{R| \text{ all } \bar{R} \notin M\}$ and $SM = \{R| \text{ if } I \lhd R \text{ then } I \notin M\}$. Note that for these and most other constructions it does not matter if $0 \in M$ or not. Thus (unless otherwise stated) we assume 0 is a member of all classes. Also recall that a class M is hereditary if $I \lhd R \in M$ implies $I \in M$.

It is well-known [see 1, p. 21] that a class Q is (Kurosh-Amitsur) *semisimple* if and only if:

(i) whenever $I \lhd R \in Q$ then $I \notin UQ$ (that is, there exists $\bar{I} \in Q$) and

(ii) if $R \notin Q$ then there exists $I \lhd R$ such that $I \in UQ$. We note that one way of saying that a class P is (Kurosh-Amitsur) *radical* is to say that $P = USP$ where SP is semisimple.

In a recent paper [2] we considered the following conditions on a class M:

(A) Every $R \in M$ has some $\bar{R} \in SUM$, and

(B) If $I \lhd R$ and some $\bar{I} \in M$ then there exists $\bar{R} \in SUM$.

We showed

Proposition 1. [2; Theorem 1, p. 219] If M is an arbitrary class, then UM is radical if and only if M satisfies (A), and UM is a hereditary radical if and only if M satisfies both (A) and (B).

In the present paper we will consider the dual problem of finding necessary and sufficient conditions on a class M so that SM will be a semisimple class, or a homomorphically closed semisimple class. It turns out, as is usually the case, that the answer is substantially simpler when the universal class (which we will designate as W) in which we assume our construction are taking place is that of all associative or alternative rings.

Another problem to which we have recently addressed ourselves [3] is that of characterizing, for a given radical P, those classes M such that $UM = P$. In the final section of the present paper we consider the dual problem of finding, for a given semisimple class Q, criteria on an arbitrary class M so that $TM=Q$. It is easy to see that classes $M \neq N$ can exist for which $SM=SN$. For example, if J is the Jacobson radical class (in the universal class W of all associative rings) and R is a simple radical ring (say, Sasiada's ring), then $SJ=SJ'$ where $J'=J \setminus (R \oplus R)$. This is clear since $SJ \subseteq SJ'$ and if $K \notin SJ$ then there exist $I \lhd K$ with $I \in J$. Now if it were possible to have $K \in SJ'$ then $I \cong R \oplus R$. But then K would have an ideal isomorphic with R contradicting $K \in SJ'$. Therefore, $K \notin SJ'$ and so $SJ=SJ'$.

It is, however, unfortunately true that the radical-semisimple duality is less than perfect, and direct dualization of the methods of [3] seem not to be helpful. Indeed, we are able to say rather less about this problem than we did [in 3] for the dual radical problem. As one example: our criteria on M that $SM=Q$, or more generally on classes M and N that $SM=SN$, require that SM and SN be hereditary classes. This is of course true in the important case of semisimple classes defined in the universal class of all associative (or alternative) rings, but the problem in the general case is open, and there are a number of other such unresolved problems.

2. *Conditions that SM be semisimple.*

The property (A) which is equivalent to UM radical has as its dual:

(1) For every $R \in M$ there exists $I \lhd R$ with $I \in USM$. However, property (1) is not equivalent to SM semisimple. In fact, we have:

Theorem 1. *Property* (1) *is a necessary but not sufficient condition that SM be a semisimple class.*

Proof. To prove necessity, suppose SM is semisimple and let $0 \neq R \in M$. Then certainly $R \notin SM$ so R has a non-zero radical $I \in USM$.

To show that (1) is not sufficient let $M=\{Z^0, Z_k^0\}$ for $k=2, 3, \ldots$, where Z^0, Z_k^0 are the zero rings on the additive groups of Z, Z_k. Clearly M satisfies (1) and we have $J \lhd I \lhd K$ where K is the ring of all (lower) triangular 3 by 3 matrices over the real numbers, I the ring of all 3 by 3 strictly triangular (zero diagonal) real matrices, and $J=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$ where x is real. Every image of I either contains a non-zero image of J and hence has an ideal in M, or is itself a zero ring and hence again has an ideal in M. Therefore, $I \in USM$. But all ideals of K are uncountable and so $K \in SM$. Thus K violates property (i) of semisimplicity.

In looking for conditions on a class M which are both sufficient and necessary that SM be semisimple we are led to consider:

(2) For an arbitrary ring R if every $\bar{R}$ has a non-zero ideal in M then whenever $R \lhd K$ it follows that K also has a non-zero ideal in M.

(3) If $I \lhd K$ with $I \in M$ then there exists some $J \lhd K$ such that every $\bar{J}$ has a non-zero ideal in M.

Theorem 2. *For M an arbitrary class, SM is semisimple if and only if M satisfies* (2) *and* (3).

Proof. Suppose first that M has properties (2) and (3) and let $R \in SM$. If $I \lhd R$ with $I \in USM$ then every $\bar{I}$ would have a non-zero ideal in M. But then (2) would require R to have a non-zero ideal in M contrary to $R \in SM$. Thus SM has property (i). Now if $R \notin SM$ then there exists some $I \lhd R$ with $I \in M$ so by (3) R has a non-zero ideal $J \in USM$, that is SM has property (ii).

On the other hand, if SM is semisimple and $I \lhd K$ with $I \in M$ then $K \notin SM$ so K has a non-zero radical $J \in USM$. Thus M has property (3). Also if $I \lhd K$ and every $\bar{I}$ has an ideal in M then $I \in USM$ so $K \notin SM$. Thus K must contain some non-zero ideal in M, that is M has property (2).

Recall that a subring A of a ring R is said to be *accessible* in R if $A = I_1 \lhd \lhd I_2 \dots \lhd I_n = R$ for some $n \geqq 1$. Consider the condition on M:

(4) If a ring R has an accessible subring in M then it has an ideal in M.

Proposition 2. The class M has property (4) if and only if SM is a hereditary class.

Proof. If M has property (4) and $I \lhd R \in SM$, then $I \in SM$ since otherwise R would have an accessible subring in M, namely some $J \lhd I$. If, on the other hand, SM is hereditary then a ring with no ideals in M is in SM, so any accessible subring would also be in SM, hence certainly not in M.

We also have

Proposition 3. For an arbitrary class M, property (4) implies property (2).

Proof. Let $I \lhd R$ where every image of I has an ideal in M. Then in particular I has such an ideal and hence by (4) the same is true of R.

Note that the converse is not true even in the case SM is semisimple, since there exist examples of non-hereditary semisimple classes [see 4].

We can now establish simpler criteria for the associative (or alternative) case.

Theorem 3. *If the universal class W which contains M (and relative to which SM is defined) is the class of all associative or alternative rings then SM is semisimple of and only if M satisfies* (1) *and* (4).

Proof. We use the criteria of Theorem 2 supposing first that M satisfies (1) and (4). By Proposition 3 we already have (2). By Proposition 2 we also have SM hereditary and it is well-known [1, Theorem 7.2, p. 25] that then USM is a radical class. Now let $I \lhd K$ with $I \in M$ so by (1) there is some $J \lhd I$ with $J \in USM$. Thus $USM(I) \neq 0$ and it is well-known that in the class of associative (or alternative) rings $USM(I) \lhd K$. There is thus a non-zero ideal of K in USM, that is, M satisfies property (3). Thus by Theorem 2 the class SM is semisimple.

On the other hand, SM semisimple implies property (1) by Theorem 1, and also implies (4) since it is well-known [1, Theorem 8.1, p. 29] that in the class of associative (or alternative) rings all semisimple classes are hereditary.

Corollary 1. In the class of all associative (or alternative) rings SM is semisimple if and only if it is hereditary and has property (ii).

Proof. The necessity is clear, and for sufficiency it follows from Proposition 2 that M satisfies (4). Also if $R \in M$ then $R \notin SM$ so by (ii) there exists $I \lhd R$ with $I \in USM$, that is M satisfies (1).

If condition (B) is dualized we have

(5) For a ring R if some $\bar{R}$ has a non-zero ideal in M then R has a non-zero ideal in USM.

The theorem on hereditary radicals [2, Theorem 1, p. 219] dualizes fully as

Theorem 4. *If SM is a semisimple class then SM is homomorphically closed if and only if M satisfies* (5).

PROOF. Suppose M has property (5) and $R \in SM$. If some $\bar{R} \notin SM$ then $\bar{R}$ has a non-zero ideal in M. By (5) R would also have an ideal in USM contrary to $R \in SM$. On the other hand, if R has no non-zero ideals in USM then it is in SM so the homomorphic closure of SM implies that no $\bar{R}$ can have such an ideal in M.

Remark that a class SM is a homomorphically closed semisimple class if and only if it is a radical-semisimple class [see 1, p. 166].

3. *Conditions on classes M and N so that $SM = SN$.*

For M an arbitrary class of rings define.

$$\overline{M} = \{R \mid I \lhd R \text{ for some } I \in M\}.$$

Remark that (by Proposition 2) when SM is hereditary $\overline{M}$ is the class of all rings having an accessible subring in M. From this definition it is immediate that

Proposition 4. For arbitrary classes M and N, if $SN \subseteq SM$ then $M \subseteq \overline{N}$ and $SN = SM$ if and only if $\overline{N} = \overline{M}$.

We also have

Proposition 5. If M is an arbitrary class and N has property (4) then $SN \subseteq SM$ if and only if $M \subseteq \overline{N}$.

PROOF. Proposition (4) is the necessity, so let $M \subseteq \overline{N}$ and suppose $R \notin SM$. Then there exists $I \lhd R$ with $I \in M$ and so some $J \lhd I$ with $J \in N$. Then J is accessible in R so R has an ideal in N, that is $R \notin SN$.

Corollary 2. If N has property (4) and $N \subseteq M$ then $SN = SM$ if and only if $M \subseteq \overline{N}$.

Corollary 3. If N has property (4) then $SN = SM$ if and only if: (a) SM is hereditary, (b) $M \subseteq \overline{N}$, and (c) $N \subseteq \overline{M}$.

In particular we note that if our universal class W is the class of all associative rings, then Corollaries 2 and 3 apply to any class N of simple rings.

Theorem 5. *For an arbitrary class M with property* (4) *the class $\overline{M}$ is the largest class such that $S\overline{M} = SM$.*

PROOF. Corollary 2 implies $S\overline{M} = SM$ and if $M \subseteq N$ where $SM = SN$ then by Proposition 5 we have $N \subseteq \overline{M}$.

Note that we easily have the dual

Theorem 5'. *For an arbitrary class M, if $\underline{M} = \{R \mid \text{some } \bar{R} \in M\}$, then $\underline{M}$ is the largest class for which $U\underline{M} = UM$.*

Remark that $\overline{M}$ and $\underline{M}$ can be very large. Indeed, if W is the universal class in which our constructions are taking place, then it is easy to show that: (i) $W =$

$=M\cup SM$ if and only if $\overline{M}=M$, and (ii) $W=M\cup UM$ if and only if $M=\underline{M}$. It is also easy to show that $UM=SM$ if and only if $\overline{M}=\underline{M}$.

As in the radical case, we find that there are in general no smallest classes for the S construction. We say that a class M satisfies a *smallest condition* for the S construction if N properly contained in M implies SM properly contained in SN. We have

Proposition 6. M satisfies a smallest condition if and only if for every $K\in M$ there exists some R such that $K\lhd R$ and for all $I\lhd R$ whenever $I\in M$ then $I\cong K$.

PROOF. First suppose M has a smallest condition and for an arbitrary $K\in M$ let $N=M\setminus\{K\}$. Then there exists $R\notin SM$, $R\in SN$. Thus R must have K as an ideal and since $R\in SN$ the only ideal in M it could have would be K (or ideals isomorphic to K).

For sufficiency suppose N is properly contained in M and let $K\in M$, $K\notin N$. By hypothesis there exists a ring R such that $K\lhd R$ and with no M-ideals other than those $\cong K$. Thus $R\in SN$ whereas $R\notin SM$, that is SM is properly contained in SN.

We can say considerably more if M has the property (somewhat stronger than (4)):

(4′) If $A\in M$ is an accessible subring of a ring R then $A\lhd R$.

Proposition 7. If M has property (4′) then M satisfies a smallest condition if and only if $I\lhd R$ with both $I, R\in M$ implies $I\cong R$.

PROOF. The sufficiency is clear from Proposition 6, so suppose M satisfies a smallest condition and let $I\lhd R$ with both $I, R\in M$. By Proposition 6 we can find a ring K such that $R\lhd K$ and with no M-ideals other than those isomorphic to R. But (4′) says that $I\lhd K$ and so $I\cong R$.

Corollary 4. In the universal class of all associative or alternative rings if M is either (a) any class of simple rings or (b) any class of subdirectly irreducible rings with unit, then M satisfies a smallest condition. Moreover, in either case (a) or case (b) if the class M is homorphically closed then SM is semisimple.

PROOF. The first statement follows from Proposition 7 provided M has property (4′). For case (b) this is easy to see for either associative or alternative rings, and the property is well-known for simple associative rings. For simple alternative rings property (4′) also follows from known results, but there is the following easy direct proof (due to Tim Anderson): Without loss of generality we may assume $J\lhd I\lhd R$ where J is simple and R alternative. Then $J^2=J$ so any member of J is the sum of terms of form xy where $x, y\in J$. If $r\in R$ then $(xy)r=(x, y, r)+x(yr)$ where $(x, y, r)=(xy)r-x(yr)$. But in an alternative ring $(x, y, r)=-(x, r, y)=-(xr)y+ +x(ry)$. Thus $(xy)r$ (and similarly $r(xy)$) is in J so $J\lhd R$.

The second statement follows from Theorem 3 since M homomorphically closed implies $M\subseteq USM$.

We conclude with some special cases of classes M and N for which $SM=SN$.

Proposition 8. If M satisfies property (4) then $S(M\cup USM)=SM$.

PROOF. This follows directly from Proposition 5 since $USM\subseteq\overline{M}$.

For an arbitrary class M we will define

$$DM = \{R \in M \mid I \in M \quad \text{for all} \quad I \lhd R\},$$

and call DM the *hereditary hull* of M. Note that this construction has already appeared [5, p. 68] where we showed: *If P is a radical class then the hereditary hull of P is the largest hereditary radical contained in P.* Suppose we also define

$$M' = \{R \mid R \notin SM\}, \quad \text{then}$$

Proposition 9. For M an arbitrary class of rings SM' is the hereditary hull of SM.

PROOF. If $R \in SM'$ then $R \in SM$ and if $I \lhd R$ then $I \notin M'$, that is $I \in SM$.

Corollary 5. For M an arbitrary class of rings, SM is hereditary if and only if $SM = SM'$.

Corollary 6. In the class of all associative or alternative rings SM is semisimple if and only if M has property (1) and $SM = SM'$.

References

[1] R. WIEGANDT, Radical and semisimple classes of rings, *Queen's Papers in Pure and Applied Mathematics, No.* 37, *Kingston*, (1974)

[2] PAUL O. ENERSEN and W. G. LEAVITT, The upper radical construction, *Publ. Math.* (Debrecen) **20** (1973), 219—222.

[3] W. G. LEAVITT, A note on upper radicals, *J. Australian Math. Soc.*, **22** *Ser. A* (1976), 264—273.

[4] W. G. LEAVITT and E. P. ARMENDARIZ, Nonhereditary semisimple classes, *Proc. Am. Math. Soc.*, **18** (1967), 1114—1117.

[5] W. G. LEAVITT, Radical and semisimple classes with specified properties, *Proc. Am. Math. Soc.*, **24** (1970), 680—687.

(Received June 5, 1975.)