Some remarks on valuations and subfields of given codimension in algebraically closed fields

To the memory of Andor Kertész

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If K is an algebraically closed field and L a subfield for which $[K:L]<\infty$ the Artin-Schreier theorem says that $[K:L]=1$ or 2. When $[K:L]=2$ L is a real closed ("maximal ordered") subfield of K. KNOPFMACHER and SINCLAIR [4] asked if a subfield L of the complex number field $\mathbf{C}$ such that $[\mathbf{C}:L]=2$ would be isomorphic to the real number field $\mathbf{R}$. BIALYNICKI-BIRULA [1] gave an example of such an L where $L\neq\mathbf{R}$, and subsequently studied subfields of prescribed codimension.

Using valuation theory we shall give explicit families of $2^{2^{\aleph_0}}$ distinct isomorphism types of subfields of codimension 2 in $\mathbf{C}$ and answer corresponding questions for arbitrary algebraically closed fields of characteristic 0.

Recall that a field F with a valution v is called maximally complete if the prolongation of v toany proper extension field of F has either strictly larger residue class field or strictly larger value group. V is called a rank one valuation if its value group is a subgroup of the additive group of the real numbers.

Lemma. *There exist $2^{2^{\aleph_0}}$ non-isomorphic ordered divisible subgroups of the additive (ordered) group of the real numbers* $\mathbf{R}$.

PROOF. Let $\{t_\alpha\}$ be a transcendency basis of $\mathbf{R}$ with respect to $\mathbf{Q}$. For any subset I of $\{t_\alpha\}$ let G_I be the divisible subgroup of $\mathbf{R}$ generated by 1 and $\{t_\alpha\}$, $t_\alpha\in I$. Since isomorphisms of ordered divisible subgroups of $\mathbf{R}$ are homotheties, the G_I's are readily checked to be non-isomorphic ordered groups. Since $\operatorname{Card}\{t_\alpha\}=$ $=2^{\aleph_0}$ the family $\{G_I\}$ has cardinality $2^{2^{\aleph_0}}$.

Theorem 1. *To any maximally complete rank one valuation w of the complex number field $\mathbf{C}$ for which the residue class field has characteristic 0 there corresponds a subfield K_w of codimension 2 in $\mathbf{C}$ such that the restriction of w to K_w is maximally complete. Here $K_{w_1}\neq K_{w_2}$ if w_1 and w_2 are inequivalent. If K_{w_1} and K_{w_2} are* (algebraically) *isomorphic, w_1 is equivalent to $w_2\varphi$ for a suitable automorphism φ of the complex field* $\mathbf{C}$.

PROOF. Let $\mathbf{C}$ be maximally complete with respect to w and let G be the value group. The resiude class field L is an algebraically closed field of characteristic 0. By the structure theorem for maximally complete fields (cf. SCHILLING [6]) we obtain that with respect to a suitable factor set from G to L the field $\mathbf{C}$ is analytically isomorphic to a formal power series field over L with exponents in G. Since the multiplicative group of L is divisible we may assume the factor set is identically 1. Hence $\mathbf{C}$ is analytically isomorphic to the formal power series field $L((G))$ consisting of all power series $l(x)=\sum_g l_g x^g$, $l_g \in L$, $g \in C$, where the support $\operatorname{supp}(l(x))=\{g\in G \mid l_g \neq 0\}$ is a well-ordered subset of G. Since L is algebraically closed of characteristic 0 and $\operatorname{Card}(L) \leqq 2^{\aleph_0}$, L may be taken to be an algebraically closed subfield of $\mathbf{C}$. $L \cap \mathbf{R}$ has then codimension 2 in L and $(L\cap\mathbf{R})((G))$ is a subfield of codimension 2 in $\mathbf{C}$ which is maximally complete with respect to w.

It is well known that a field is algebraically closed if it is complete with respect to two non-equivalent rank one valuations. This gives the second assertion of the theorem. Finally, if σ is an (algebraical) isomorphism from K_{w_1} to K_{w_2} K_{w_1} is complete with respect to the valuations w_1 and $w_2\sigma$ which are thus equivalent on K_{w_1}. σ extends to an automorphism φ of $\mathbf{C}\cdot w_1$ and $w_2\varphi$ are thus prolongations of valuations which are equivalent on K_{w_1}. Since K_{w_1} in particular is henselian, w_1 and $w_2\varphi$ are equivalent on $\mathbf{C}$.

Corollary. There exist exactly $2^{2^{\aleph_0}}$ *non-isomorphic (non-archimedean ordered) subfields of codimension* 2 *in* $\mathbf{C}$.

PROOF. For any ordered divisible subgroup G of $\mathbf{R}$ there exists a valuation of $\mathbf{C}$ with value group G for which $\mathbf{C}$ is maximally complete and the corresponding residue class field has characteristic 0. By the lemma there are $2^{2^{\aleph_0}}$ such groups G and the corresponding subfields of $\mathbf{C}$ are non-isomorphic by theorem 1 (and readily checked to be non-archimedean ordered). This proves the corollary since obviously there cannot be more than $2^{2^{\aleph_0}}$ subfields of $\mathbf{C}$.

Remark 1. It is easy to see that there are $2^{2^{\aleph_0}}$ non-isomorphic archimedean ordered subfields of codimension 2 in $\mathbf{C}$. The intersection of all such subfields is the field of all totally real algebraic numbers.

Remark 2. The above proofs show that if G runs through $2^{2^{\aleph_0}}$ non-isomorphic subgroups of R the power series fields $\mathbf{R}((G))$ form $2^{2^{\aleph_0}}$ distinct isomorphism types of fields embeddable with codimension 2 in $\mathbf{C}$.

Remark 3. An explicit family of non-isomorphic fields of codimension 2 in $\mathbf{C}$ can be obtained as follows: If $L_0=R$, $L_{i+1}=\bigcup_{n=1}^{\infty} L_i((x^{\frac{1}{n}}))$, $i \geqq 0$, then the fields L_i are non-isomorphic and embeddable with codimension 2 in $\mathbf{C}$. Each L_i is henselian but not complete with respect to any valution.

Next we consider arbitrary algebraically closed fields of characteristic 0.

Theorem 2. *Let* K *be an algebraically closed field of characteristic* 0. *If* K *is an algebraic extension of the rational number field* $\mathbf{Q}$ *there is up to isomorphism just one subfield of codimension* 2 *in* K. *Otherwise there are exactly* $2^{\operatorname{Card}(K)}$ *non-isomorphic subfields of condimension* 2 *in* K. *Moreover, for any subfield* L *of codimension* 2 *in* K *there are* $2^{\operatorname{Card}(K)}$ *isomorphic subfields of codimension* 2 *in* K.

Proof. Let t be the transcendency degree of K with respect to $\mathbf{Q}$. If $t=0$ any real closed subfield of K is isomorphic to the field of all real algebraic numbers, and the restriction to any such subfield of the $2^{\aleph_0}$ automorphisms of K give rise to $2^{\aleph_0}$ isomorphic copies.

If $t>0$ we first consider the case where t is finite. Let $\{\alpha_i\}$, $(\mathrm{Card}(\{\alpha_i\})=2^{\aleph_0})$ be a transcendency basis for the real number field $\mathbf{R}$ over $\mathbf{Q}$. $\{\alpha_i\}$ contains $2^{\aleph_0}$ mutually disjoint subsets A_j each consisting of t elements. Let K_j be the real closure of $\mathbf{Q}\{\alpha_\nu\}$, $(\alpha_\nu\in A_j)$ in $\mathbf{R}$ with respect to the order induced by $\mathbf{R}$. The ordered fields K_j are obviously pairwise non-isomorphic. The algebraic closure of each K_j is an extension of degree 2 and is isomorphic to K; hence the first statement of the theorem is proved when t is finite. As for the second statement let L be a subfield of condimension 2 in K. Let $\beta_1, \ldots, \beta_t$ be a transcendency basis for $L/\mathbf{Q}$. There is an automorphism σ of $\mathbf{Q}(\beta_1, \ldots, \beta_t)$ which doest not preserve the order that $L(\supseteqq \mathbf{Q}(\beta_1, \ldots, \beta_t))$ has. σ extends to $2^{\aleph_0}$ distinct automorphism of $K/\mathbf{Q}$ whose restrictions to L give rise to $2^{\aleph_0}$ isomorphic copies of L in K.

Next consider the case where t is an infinite cardinal number $\aleph$.

Clearly $\mathrm{Card}(K)=\aleph$. Let G be the direct sum

$$G=\sum_{\alpha<\aleph}\oplus A_\alpha e_\alpha, \quad A_\alpha=\mathbf{Z} \quad \text{or} \quad \mathbf{Z}+\mathbf{Z}\sqrt{2}$$

where α runs through all ordinal numbers $<\aleph$. We order G lexicographically. Let $F=\mathbf{Q}((G))$ be the field of all formal power series on G over $\mathbf{Q}$. Any element $k\in F$ can be written $k=\sum_g k_g x^g$, $k_g\in\mathbf{Q}$ where the support of k $\mathrm{supp}(k)=\{g\in G \mid k_g\neq 0\}$ is a wellordered subset of G. F can be ordered in the obvions lexicographic way.

Generally, to any (non-archimedean) ordering of a field corresponds a valuation whose valuation ring consits of all elements which are not infinitely greater than 1 (cf. [2] Ex. 3 p. 171).

For $F=\mathbf{Q}((G))$ this valuation w_G is defined by $w_G(k)=\min\{g\}$, $(g\in\mathrm{supp}(k))$. Let $B=\{\alpha \mid A_\alpha=Z\}$, $C=\{\alpha \mid A_\alpha=\mathbf{Z}+\mathbf{Z}\sqrt{2}\}$ and $D=\bigcup_{\alpha\in B} e_\alpha \cup \bigcup_{\alpha\in C} e_\alpha \cup \bigcup_{\alpha\in C}\sqrt{2}e_\alpha$. Obviously $\mathrm{Card}(D)=\aleph$. Let $\{u_d\}$, $d\in D$ be a transcendency basis for K over $\mathbf{Q}$ and set $M=\mathbf{Q}(\{u_d\})$. Define a mapping φ from M to F by $\varphi(u_d)=X^d$ and $\varphi(q)=q$, $q\in\mathbf{Q}$. Hereby one gets an isomorphism of M onto a subfield of F and thus and ordering of M. The corresponding valuation is $w_G\varphi$. G is the value group of w_G. Let M_G be a real closure in K of the ordered field M. M_G is an ordered field whose corresponding valuation has the value group $G\otimes\mathbf{Q}=\sum_{\alpha<\aleph}\oplus(A_\alpha\otimes\mathbf{Q})l_\alpha$ where $A_\alpha\otimes\mathbf{Q}=\mathbf{Q}$ or $\mathbf{Q}+\mathbf{Q}\sqrt{2}$ according as $\alpha\in B$ or $\alpha\in C$.

In the definition of G we have for each α two choise for A_α and we thus get $2^{\aleph}$ ordered groups. By considering skeletons (cf. Fuchs [3] or Ribenboim [5]) it is easily seen that the corresponding groups $G\otimes\mathbf{Q}$ are pairwise non-isomorphic (qua ordered groups). This implies that the corresponding real closed fields M_G are non-isomorphic since any isomorphism would be order-preserving; the corresponding valuations w_G would then be equivalent and in particular have order-isomorphic value groups.

Finally, let L be a subfield of codimension 2 in K. If $\{\beta_i\}$ is a transcendency basis for L over $\mathbf{Q}$ there are $2^{\aleph}$ permutations of $\{\beta_i\}$ which extend to $2^{\aleph}$ automorphisms of $K/\mathbf{Q}$. The restrictions of these automorphisms to L give rise to $2^{\aleph}$ distrinct isomorphic sopies of L having codimension 2 in K. Theorem 2 is now proved.

Remark. It is not hard to show that the intersection of all subfields of codimension 2 in K is the field of all totally real algebraic numbers (i.e. algebraic numbers all of whose conjugates are real). One could also mention, that a subfield of codimension 2 in K has a trivial automorphism group if and only if it is archimedian ordered. In fact, any nonarchimedian ordered real closed field L has at least Card (L) distinct automorphisms.

Bialynicki-Birula [1] proved that any algebraically closed field has a subfield of countable condimension. We shall finish by proving the following

Theorem 3. *Let K be an arbitrary algebraically closed field of characteristic* 0. *If* Card $(K)\geqq 2^{\aleph_0}$ *and* $\aleph$ *an infinite cardinal number* $<$ Card (K) *there exists a subfield L of K for which* $[K:L]=\aleph$.

Corollary. Assuming the continuum hypothesis an algebraical closed field of characteristic 0 has subfield of any prdscribed infinite codimension $<$ Card (K).

Proof. (of Theorem 3). Let Card $(K)=\overline{\aleph}$ and let G be the lexicographically ordered vector space over the rational field $\mathbf{Q}$:

$$G = \sum_{\alpha<\overline{\aleph}} \oplus \mathbf{Q}e_\alpha$$

where α runs through all ordinals $<\overline{\aleph}$. For any finite set I of ordinals $<\overline{\aleph}$ let F_I be the subfield of the formal power series field $\mathbf{C}((G))$ consisting of all power series of the form $\sum_g c_g x^g$ $c_g\in\mathbf{C}$ where the exponents g belong to the $\mathbf{Q}$-space generated by $\{e\alpha\}$, $\alpha\in I$ and the denominators of q_α in the expression $g=\Sigma g_\alpha e_\alpha$, $q_\alpha\in\mathbf{Q}$, are bounded.

$\mathbf{C}((G))$ is a maximally complete field with algebraically closed residue class field and divisible value group and hence algebraically closed. F_I is algebraically closed in $\mathbf{C}((G))$ and consequently itself an algebraically closed field.

For an arbitrary set I of ordinals $<\overline{\aleph}$ we define $F_I=\bigcup_{I'} F_{I'}$ where I' runs through all finite subsets of I. Further we define F_I^* as the subfield of F_I of all power series whose exponents g have "integral coefficients", i.e. belong to $\sum_{\alpha<\overline{\aleph}} \oplus \mathbf{Z}e_\alpha$.

Now let J be the set of all ordinals $<\overline{\aleph}$. Then Card$(F_j)=$Card $(J)=\overline{\aleph}$ since $\overline{\aleph}>2^{\aleph_0}$. Hence $F_j\simeq K$. We now choose a subset I of J such that Card $(J\setminus I)=\aleph$ and set $L=F_J^*\cap F_I$. The elements $x^g, g=\Sigma\, q_\alpha e_\alpha$ (finite sum), $\alpha\in J\setminus I$ where the q_α's run through a set of representatives of the rational numbers modulo the integers, form a basis for F_J over L. Thus $[F_J:L]=\aleph$ and the theorem is proved in view of the isomorphism $F_J\simeq K$.

References

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(Received July 1, 1975.)