On generalized absolute Cesaro summability factors

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Let Σa_n be a given series with partial sums s_n. By $\sigma_n^{(\alpha)}$ we denote the n-th Cesaro means of order α of the sequences $\{s_n\}$. If the series $\Sigma n^{k-1}|\sigma_n^{(\alpha)}-\sigma_{n-1}^{(\alpha)}|^k$ $(k\geqq 1; \alpha>-1)$ is convergent, we say that the series Σa_n is absolute summable (C, α) with index k or simply summable $|C, \alpha|_k$.

A sequence $\{\lambda_n\}$ is said to be a $|C, \alpha|_k$ summability factor of the series Σa_n if multiplying it term by term with λ_n, the factored series is $|C, \alpha|_k$ summable.

We have the following theorems.

Theorem 1. *Let $k\geqq 1, 0\leqq\alpha\leqq 1$ and let $\{\lambda_n\}$ be a positive non-increasing and convex sequence, $\{\gamma_n\}$ be a positive non-decreasing sequence such that $\Delta\gamma_\mu=O(\gamma_n/n)$ $(n=1, 2, \ldots; \mu=n, n+1, \ldots, 2n)$. If*

$$\sum_{n=1}^{\infty} \lambda_n^k|\Delta\gamma_n| < \infty \tag{1}$$

and

$$\sum_{\nu=1}^{n} \nu^{k-1}|\sigma_\nu^{(\alpha)}-\sigma_{\nu-1}^{(\alpha)}|^k = O(\gamma_n) \tag{2}$$

then the sequence $\{\lambda_n\}$ is a $|C, \alpha|_k$ summability factor of the series Σa_n.

The special case $k=1, \gamma_n=\log n$ and $\lambda_n=(\log n)^{-1-\varepsilon}$ $(\varepsilon>0)$ was proved by SUNOUCHI [7]. Using a result of NÉMETH (see Lemma 2) we can prove that

$$\sum_{\nu=1}^{n} \nu^{k-1}|\sigma_\nu^{(\alpha)}-\sigma_{\nu-1}^{(\alpha)}|^k = O\left(\sum_{\nu=1}^{n} \frac{|s_\nu|^k}{\nu^{k(\alpha-1)+1}}\right),$$

so we have the following.

Corollary. *If the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ are satisfying the conditions of Theorem 1 and*

$$\sum_{\nu=1}^{N} \frac{|s_\nu|^k}{\nu^{k(\alpha-1)+1}} = O(\gamma_n)$$

then the series $\Sigma\lambda_n a_n$ is $|C, \alpha|_k$ summable.

The special case $\alpha=1$ and $\gamma_n=\log n$ was proved by MAZHAR [3] and MISHRA [4].

The next theorem gives a result concerning the $|C,\alpha|_k$ summability of negative order α, too.

Theorem 2. *Let* $k\geqq 1$; $-\dfrac{1}{k}<\alpha\leqq 1$ *and let* $\{\lambda_n\}$ *be a positive convex sequence such that*

$$\sum_{n=1}^{\infty}\frac{\lambda_n^k}{n^{k(\alpha-1)+1}}<\infty. \tag{3}$$

If

$$\sum_{\nu=1}^{n}\left|\frac{1}{\nu}\sum_{\mu=1}^{\nu}\mu a_\mu\right|^k=O(n) \tag{4}$$

then the sequence $\{\lambda_n\}$ *is a* $|C,\alpha|_k$ *summability factor of the series* Σa_n.

Finally we have two results for Fourier series.

Theorem 3. *If* $\{\lambda_n\}$ *satisfies the conditions of Theorem* 2 *then* $\{\lambda_n\}$ *is a* $|C,\alpha|_k$ *summability factor of Fourier series of any integrable function almost everywhere.*

The special case $k=1$ and $\alpha=1$ was proved by Chow [1] and Sunochi [6]. For $f(x)\in L^p$ we define

$$\omega_p(t,f)=\sup_{0<h\leqq t}\left\{\int_{-\pi}^{\pi}|f(x+h)+f(x-h)-2f(x)|^p\,dx\right\}^{1/p}.$$

Theorem 4. *Let* $f(x)\in L^p$ $(1<p\leqq 2)$, $1\leqq k<2$ *and let* $\{\mu_n\}$ *be a non-increasing sequence tending to zero and satisfying the condition*

$$\sum_{n=1}^{\infty}\frac{1}{n\left(\sum_{\nu=1}^{n}\mu_\nu\right)^{\frac{2k}{2-k}}}<\infty.$$

If

$$\sum_{n=1}^{\infty}\mu_n\omega_p\left(\frac{1}{n},f\right)<\infty$$

then the Fourier series of $f(x)$ *is summable* $|C,\alpha|_k$ *almost everywhere for* $\alpha>\dfrac{1}{p}$.

The special case $k=1$ was proved in [8]. The proof of Theorem 4 runs similarly to that of Theorem I of [8], we only have to use Lemma 1 instead of Lemma 4 of [8], so we omit the proof of Theorem 4.

We need the following lemmas. Generalizing and a little lit modifying a *Chow's lemma* ([2]. Lemma 4) we obtain

Lemma 1. *Let* $k\geqq 1, 0\leqq\alpha\leqq 1$ *and let* $\{\lambda_n\}$ *be a positive sequence such that the sequence* $\{|\Delta\lambda_n|\}$ *is*[1]) *non-increasing. If*

$$\sum_{n=1}^{\infty}\frac{\lambda_n^k|t_n^{(\alpha)}|^k}{n}<\infty\quad\left(t_n^{(\alpha)}=n(\sigma_n^{(\alpha)}-\sigma_{n-1}^{(\alpha)})\right) \tag{5}$$

[1]) $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$

and

(6) $$\sum_{n=1}^{\infty} n^{k-1}|\Delta\lambda_n|^k\,|t_n^{(\alpha)}|^k < \infty$$

then[2]) *the series* $\Sigma\,\lambda_n a_n$ *is* $|C, \alpha|_k$ *summable.*

We require the following inequality of NÉMETH ([5]) as

Lemma 2. *Let* $d_n \geqq 0$ *and* $l_n > 0$ $(n=1, 2, \ldots)$ *be given sequences. If the triangular matrix* $M=(C_{n,\nu})$ *satisfies the following conditions* $C_{n,\nu}\geqq 0$ $(\nu<n)$, $C_{n,\nu}=0$ $(\nu\geqq n)$ *and* $0\leqq C_{m,\nu}\leqq C\cdot C_{n,\nu}$ $(0<\nu\leqq n\leqq m)$ C *denotes a positive absolute constant, then for any* $k\geqq 1$

$$\sum_{n=1}^{\infty} l_n\left(\sum_{\nu=1}^{n} C_{n,\nu}\, d_\nu\right)^k \leqq C^{k(k-1)}k^k \sum_{n=1}^{\infty} l_n^{1-k}\left(\sum_{\nu=n}^{\infty} l_\nu C_{\nu,n}\right)^k d_n^k.$$

Using the Abel transformation it is easy to prove the following lemmas.

Lemma 3. *If* $k\geqq 1$ *and* $\{b_n\}$ *is a non-increasing sequence such that* $\sum_{n=1}^{\infty} b_n<\infty$, *then the series* $\sum_{n=1}^{\infty} n^{k-1}b_n^k$ *is convergent and for every* $\nu=1, 2, \ldots$

(7) $$\sum_{n=4\nu+1}^{\infty} n^{k-1}b_n^k = O\left(\left\{\sum_{n=\nu}^{\infty} b_n\right\}^k\right)$$

hold.

Lemma 4. *Let* $\{b_n\}$ *be a non-increasing sequence such that* $\sum_{n=1}^{\infty} b_n<\infty$ *and* $\{C_n\}$ *be a sequence of positive numbers such that* $nC_\mu=O\left(\sum_{i=1}^{n} c_i\right)$, *where* $n=1, 2, \ldots$; $\mu=n, n+1, \ldots, 2n$. *If* $k\geqq 1$ *and*

(8) $$\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} b_m\right)^k C_n < \infty$$

then the series $\sum_{n=1}^{\infty} n^k b_n^k C_n$ *is convergent.*

Finally we shall use the following obvious facts as lemmas.

Lemma 5. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\varrho_n\}$ *be given positive sequences such that* $\{\alpha_n\}$ *non-increasing,* $\{\varrho_n\}$ *non-decreasing. If* $\sum_{\nu=1}^{n}\alpha_{\nu+1}|\Delta\varrho\nu|=O(1)$ *and* $\sum_{\nu=1}^{n}\beta_\nu=O(\varrho_n)$ *then* $\sum_{\nu=1}^{n}\alpha_\nu\beta_\nu=O(1)$.

[2]) If $\alpha=0$ the condition (6) is superfluous.

Lemma 6. *Let* $\alpha_n \geqq 0$ *and* $\beta_n \geqq 0$ *be given sequences. If* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $C_1 \alpha_{2^{n+1}} \leqq \leqq \alpha_{2^n+\nu} \leqq C_2 \alpha_{2^n}$ *and* $\sum_{\mu=1}^{n} \beta_\mu = O(n)$ $(n=1, 2, \dots; \nu=1, 2, \dots, 2^n)$ *then* $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$.

PROOF *of Theorem* 1.

First using Lemma 5 with $\alpha_n = \lambda_n^k$, $\beta_n = n^{k-1} |\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^k$ and $\varrho_n = \gamma_n$ by (1) and (2) we obtain (5).

Secondly we use Lemma 5 $\alpha_n = (\Delta \lambda_n)^k$, $\beta_n = n^{2k-1} |\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^k$ and $\varrho_n = n^k \gamma_n$. By (2) we have that $\sum_{\nu=1}^{n} \beta_\nu = O(\varrho_n)$. If we can show that

$$\sum_{n=1}^{\infty} (\Delta \lambda_{n+1})^k \, |\Delta (n^k \gamma_n)| < \infty \tag{9}$$

then, by Lemma 5, we have (6). Finally we apply Lemma 1 and have Theorem 1.

Now we prove (9)

$$\sum_{n=1}^{\infty} (\Delta \lambda_{n+1})^k \, |\Delta (n^k \gamma_n)| \leqq K(k) \sum_{n=1}^{\infty} n^k (\Delta \lambda_n)^k \, |\Delta \gamma_n| +$$

$$+ K(k) \sum_{n=1}^{\infty} n^{k-1} (\Delta \lambda_n)^k \gamma_n \equiv S_1 + S_2.$$

The constant K is depending on its argument only.

Using Lemma 4 with $b_n = \Delta \lambda_n$ and $C_n = |\Delta \gamma_n|$, by (1) and (8) we have that S_1 is finite.

Applying Lemma 3 with $b_n = \Delta \lambda_n$ and $C_n = |\Delta \gamma_n|$ we get

$$S_2 \leqq K(k) \left(\sum_{n=2}^{\infty} n^{k-1} (\Delta \lambda_n)^k \sum_{\nu=1}^{n-1} |\Delta \gamma_\nu| + \gamma_1 \sum_{n=2}^{\infty} n^{k-1} (\Delta \lambda_n)^k \right) =$$

$$= K(k) \left(\sum_{\nu=1}^{\infty} |\Delta \gamma_\nu| \sum_{n=\nu+1}^{\infty} n^{k-1} (\Delta \lambda_n)^k + O(1) \right) =$$

$$= K(k) \left(\sum_{\nu=1}^{\infty} |\Delta \gamma_\nu| \sum_{n=\nu+1}^{4\nu} n^{k-1} (\Delta \lambda_n)^k + \sum_{\nu=1}^{\infty} |\Delta \gamma_\nu| \sum_{n=4\nu+1}^{\infty} n^{k-1} (\Delta \lambda_n)^k \right) \equiv$$

$$\equiv S_{21} + S_{22}.$$

Since $\{\Delta \lambda_n\}$ is non-increasing we have $S_{21} = O(S_1)$.

Applying Lemma 3 again, by (7) and (1), we can see that S_{22} is also finite so, our proof is complete.

PROOF *of Theorem* 2. Let us denote by $\sigma_n^{(\alpha)}(\lambda)$ the n-th (C, α) mean of the series $\Sigma \lambda_n a_n$. Using the symbols

$$A_n^{(\alpha)} = \binom{n+\alpha}{n} \quad \text{and} \quad L_{n,\nu}^{(\alpha)} = \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} \cdot \frac{\nu \alpha}{(n+1-\nu)(n+1+\alpha)} \quad (\nu = 1, \dots, n),$$

by Abel's transformation we have that for $n \geqq 3$

$$\sigma_n^{(\alpha)}(\lambda) - \sigma_{n-1}^{(\alpha)}(\lambda) = \sum_{\nu=1}^{n-1} L_{n-1,\nu}^{(\alpha)} \lambda_\nu a_\nu + \frac{1}{A_n^{(\alpha)}} \lambda_n a_n = \tag{10}$$

$$= \sum_{\nu=1}^{n-2} \left(\sum_{\mu=1}^{\nu} \mu a_\mu\right) \left(\frac{L_{n-1,\nu}^{(\alpha)}}{\nu} \lambda_\nu - \frac{L_{n-1,\nu+1}^{(\alpha)}}{\nu+1} \lambda_{\nu+1}\right) +$$

$$+ \left(\sum_{\mu=1}^{n-1} \mu a_\mu\right) \left(\frac{L_{n-1,n-1}^{(\alpha)}}{n-1} \lambda_{n-1} - \frac{\lambda_n}{n A_n^{(\alpha)}}\right) + \left(\sum_{\nu=1}^{n} \mu a_\mu\right) \frac{\lambda_n}{n A_n^{(\alpha)}}.$$

A simple computation shows that

(11)

$$\frac{L_{n-1,\nu}^{(\alpha)}}{\nu} \lambda_\nu - \frac{L_{n-1,\nu+1}^{(\alpha)}}{\nu+1} \lambda_{\nu+1} = \frac{\alpha}{(n+\alpha) A_{n-1}^{(\alpha)}} \left(\frac{A_{n-\nu-1}^{(\alpha)}}{n-\nu} \Delta\lambda_\nu + \frac{(\alpha-1) A_{n-\nu-2}^{(\alpha)}}{(n-\nu+1)(n-\nu)} \lambda_{\nu+1}\right)$$

and

$$\frac{L_{n-1,n-1}^{(\alpha)}}{n-1} \lambda_{n-1} - \frac{\lambda_n}{n A_n^{(\alpha)}} = \frac{1}{(n+\alpha) A_{n-1}^{(\alpha)}} (\alpha \Delta\lambda_{n-1} + (\alpha-1)\lambda_n). \tag{12}$$

Looking at (10), (11) and (12) we get the estimation

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^{(\alpha)}(\lambda) - \sigma_{n-1}^{(\alpha)}(\lambda)|^k \leqq \tag{13}$$

$$\leqq C(\alpha, k) \sum_{n=2}^{\infty} \frac{1}{n^{k\alpha+1}} \left(\sum_{\nu=1}^{n-1} (n-\nu)^{\alpha-1} \left|\sum_{\mu=1}^{\nu} \mu a_\mu\right| s\lambda_\nu\right)^k +$$

$$+ C^*(\alpha, k) \sum_{n=3}^{\infty} \frac{1}{n^{k\alpha+1}} \left(\sum_{\nu=1}^{n-2} (n-\nu-1)^{\alpha-2} \left|\sum_{\mu=1}^{\nu} \mu a_\mu\right| \lambda_{\nu+1}\right)^k +$$

$$+ C(\alpha, k) \sum_{n=1}^{\infty} \frac{1}{n^{k\alpha+1}} \left|\sum_{\mu=1}^{n} \mu a_\mu\right|^k \lambda_n^k \equiv S_3 + S_4 + S_5.$$

First we shall consider S_5. Since

$$S_5 = C(\alpha, k) \sum_{n=1}^{\infty} \frac{\lambda_n^k}{n^{k(\alpha-1)+1}} \left|\frac{1}{n} \sum_{\mu=1}^{n} \mu a_\mu\right|^k,$$

by (3), (4) and Lemma 6 we get that S_5 is finite.

We shall now estimate S_3. We put $C_{n,\nu} = (n-\nu)^{\alpha-1}$ for $\nu < n$, $d_n = |\sum_{\mu=1}^{n} \mu a_\mu| \Delta\lambda_n$, $l_n = n^{-k\alpha-1}$ and applying Lemma 2 we have

$$S_3 \leqq C(\alpha, k) \sum_{n=2}^{\infty} \left(\frac{1}{n^{k\alpha+1}}\right)^{1-k} \left(\sum_{\nu=n+1}^{\infty} \frac{(\nu-n)^{\alpha-1}}{\nu^{k\alpha+1}}\right)^k \left|\sum_{\mu=1}^{n} \mu a_\mu\right|^k (\Delta\lambda_n)^k$$

and from this a standard computation shows that

$$S_3 \leqq C(\alpha, k) \sum_{n=2}^{\infty} \frac{(\Delta\lambda_n)^k}{n^{k(\alpha-1)+1}} \left|\sum_{\mu=1}^{n} \mu a_\mu\right|^k.$$

If we can show that the series

$$\sum_{n=1}^{\infty} \frac{(\Delta\lambda_n)^k}{n^{k(\alpha-2)+1}}$$

is convergent, then by (4) and Lemma 6 we get S_3 is finite. By condition (3) we have

$$\sum_{n=5}^{\infty} \frac{(\Delta\lambda_n)^k}{n^{k(\alpha-2)+1}} \leqq \sum_{m=2}^{\infty} (2^{m+1}\Delta\lambda_{2^m})^k \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n^{k(\alpha-1)+1}} \leqq$$

$$\leqq C(k) \sum_{m=2}^{\infty} \lambda_{2^{m-1}}^k \sum_{n=2^{m-2}+1}^{2^{m-1}} \frac{1}{n^{k(\alpha-1)+1}} \leqq C(k) \sum_{n=1}^{\infty} \frac{\lambda_n^k}{n^{k(\alpha-1)+1}} < \infty.$$

It remains to estimate S_4. Considering (10) and (11) we can choose $C^*(1, k)=0$ in (13), so we can assume that $-\frac{1}{k}<\alpha<1$. Putting $C_{n,\nu}=(n-\nu-1)^{\alpha-2}$ for $\nu<n-1$ and zero for $\nu=n-1$, $d_n=|\sum_{\mu=1}^{n} \mu a_\mu|\lambda_{n+1}$ $l_n=n^{-k\alpha-1}$ and applying Lemma 2 we have

$$S_4 \leqq C(\alpha, k) \sum_{n=3}^{\infty} \left(\frac{1}{n^{k\alpha+1}}\right)^{1-k} \left(\sum_{\nu=n+2}^{\infty} \frac{(\nu-n-1)^{\alpha-2}}{\nu^{k\alpha+1}}\right)^k \left|\sum_{\mu=1}^{n} \mu a_\mu\right|^k \lambda_{n+1}^k \leqq$$

$$\leqq C(\alpha, k) \sum_{n=3}^{\infty} \frac{1}{n^{k\alpha+1}} \left|\sum_{\mu=1}^{n} \mu a_\mu\right|^k \lambda_{n+1}^k \leqq C(\alpha, k)\, S_5.$$

Our proof is complete.

PROOF *of Theorem* 3. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{14}$$

and let $s_n(x)$ and $\sigma_n(x)$ denote the n-th partial sum and n-th $(C, 1)$ mean of the Fourier series (14), respectively. We have by *Zygmond's theorem* ([9], vol. II. p. 184.) that

$$\sum_{\nu=1}^{n} |s_\nu(x)-\sigma_\nu(x)|^k \leqq C(x, k)\cdot n$$

almost everywhere. Using the identity

$$s_\nu(x)-\sigma_\nu(x) = \frac{1}{\nu+1} \sum_{\mu=1}^{\nu} \mu A_\mu(x)$$

by Theorem 2 we get Theorem 3.

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